

# Lagrangian tetragons and instabilities in Hamiltonian dynamics

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## Abstract

We present a new existence mechanism, based on symplectic topology, for orbits of Hamiltonian flows connecting a pair of disjoint subsets in the phase space. The method involves function theory on symplectic manifolds combined with rigidity of Lagrangian submanifolds. Applications include superconductivity channels in nearly integrable systems and dynamics near a perturbed unstable equilibrium.

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# 1 Introduction and main results

Given a Hamiltonian flow and a pair of disjoint subsets in the phase space, does there exist an orbit connecting these subsets? Various instances of this question arise in the study of instabilities in Hamiltonian dynamics. In this paper we combine two seemingly remote approaches to detecting such orbits:

- the one based on the Poisson bracket invariants coming from function theory on symplectic manifolds [4];
- a geometric construction due to Mohnke [19], yielding what we call below a *Lagrangian tetragon*, which enabled him at the time to confirm a version of Arnold's chord conjecture in Reeb dynamics.

The obtained package turns out to be efficient in a number of specific models, including superconductivity channels in nearly integrable systems and dynamics near a perturbed unstable equilibrium that will be discussed below. The proposed existence mechanism for connecting orbits is robust with respect to perturbations of the Hamiltonian in the uniform norm.

## 1.1 Interlinking: a chord existence mechanism

SETTING THE STAGE. Given an arbitrary smooth (possibly time-dependent) vector field  $v$  on a smooth manifold, the piece of its integral trajectory defined over a time interval  $[t_0, t_1]$ ,  $t_0 < t_1$ , is called a *chord of  $v$* , or a *chord of the flow of  $v$* , of *time-length*  $t_1 - t_0$ . If such a chord passes at the time  $t_0$  through a set  $X_0$  and at the time  $t_1$  through a set  $X_1$  we say that it is a *chord from  $X_0$  to  $X_1$* .

If the flow of a vector field is defined everywhere for all times, we say that the vector field is *complete*.

Let  $(M, \omega)$  be a connected symplectic manifold. Consider a smooth time-periodic Hamiltonian  $G : M \times \mathbb{S}^1 \rightarrow \mathbb{R}$ , where  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ . Given a pair of disjoint compact subsets  $X_0, X_1 \subset M$ , a (*Hamiltonian*) *chord* of  $G$  from  $X_0$  to  $X_1$  is a chord from  $X_0$  to  $X_1$  of the Hamiltonian vector field defined by  $G$ . If this vector field is complete, we say that  $G$  is *complete*.

SEPARATION. A function  $G \in C^\infty(M \times \mathbb{S}^1)$   $\Delta$ -separates two disjoint compact sets  $Y_0, Y_1 \subset M$  if

$$\Delta = \Delta(G; Y_0, Y_1) := \min_{Y_1 \times \mathbb{S}^1} G - \max_{Y_0 \times \mathbb{S}^1} G > 0 .$$

Note that in this definition the order of sets  $Y_0$  and  $Y_1$  is important:  $G$  is larger on  $Y_1$  than on  $Y_0$ .

INTERLINKING.

**Definition 1.1.** Let  $(X_0, X_1), (Y_0, Y_1)$  be two pairs of disjoint sets:  $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$ .

We say that the pair  $(Y_0, Y_1)$   $\kappa$ -interlinks the pair  $(X_0, X_1)$ ,  $\kappa > 0$ , if every complete (time-dependent) Hamiltonian which  $\Delta$ -separates  $Y_0$  and  $Y_1$  admits a chord from  $X_0$  to  $X_1$  of time-length  $\leq \kappa/\Delta$ .

If this property is known to hold only for complete *autonomous* Hamiltonians on  $M$ , we say that the pair  $(Y_0, Y_1)$  *autonomously*  $\kappa$ -interlinks the pair  $(X_0, X_1)$ .

Interlinking is the central phenomenon discussed in the present paper. The existence of interlinking pairs is non-obvious and will be discussed below. The interlinking of  $(X_0, X_1)$  and  $(Y_0, Y_1)$  can be formulated in terms of a Poisson bracket invariant of the quadruple  $(X_0, X_1, Y_0, Y_1)$  – see Section 1.5.

We will also discuss a stable version of the interlinking phenomenon. Namely, given a closed connected manifold  $N$ , define the  $N$ -stabilization of a subset  $X \subset M$  as is its product  $X \times N \subset M \times T^*N$  with the zero-section  $N$  of  $T^*N$  (the trivial case  $N = T^*N = \{\text{a point}\}$  is also allowed; thus, a set can be viewed as a trivial stabilization of itself). We say that a pair  $(Y_0, Y_1)$  *stably*  $\kappa$ -interlinks the pair  $(X_0, X_1)$  if for all  $m \in \mathbb{Z}_{\geq 0}$  the pair  $(Y_0 \times \mathbb{T}^m, Y_1 \times \mathbb{T}^m)$   $\kappa$ -interlinks the pair  $(X_0 \times \mathbb{T}^m, X_1 \times \mathbb{T}^m)$ , where  $\mathbb{T}^m$  is the  $m$ -dimensional torus.

ROBUSTNESS. Observe that the existence of a Hamiltonian chord of  $G$  provided by the interlinking is robust with respect to perturbations of  $G$  for

which the perturbed Hamiltonian  $G + F$  is complete and the perturbation term  $F$  is sufficiently small on  $Y_0 \cup Y_1$ . Indeed,

$$\Delta(G + F; Y_0, Y_1) \geq \Delta(G; Y_0, Y_1) - |\Delta(F; Y_0, Y_1)|. \quad (1)$$

Therefore if  $(Y_0, Y_1)$   $\kappa$ -interlinks  $(X_0, X_1)$  and  $G$   $\gamma$ -separates  $X_0$  and  $X_1$ , the perturbed Hamiltonian  $G + F$  has a chord from  $X_0$  to  $X_1$  of time-length  $\leq \kappa/(\gamma - \delta)$ , where  $\delta := |\Delta(F; Y_0, Y_1)|$  is assumed to be less than  $\gamma$ .

Let us emphasize that the perturbation  $F$  can be arbitrarily large away from  $Y_0 \cup Y_1$  and can have large derivatives everywhere – thus the dynamics generated by  $G + F$  might be completely different from the dynamics generated by  $G$  and the chord of  $G + F$  does not have to be close in any sense to the chord of  $G$ .

## 1.2 Introducing Lagrangian tetragons

Next, we illustrate the notion of interlinking for a special class of examples which plays a key role in dynamical applications discussed further in the paper. The following construction originates in the work of Mohnke [19] (for another application of Mohnke’s construction to Hamiltonian dynamics see [18]).

Let  $(\Sigma^{2k-1}, \xi)$ ,  $k \geq 1$ , be a (not necessarily closed) contact manifold with a co-orientable contact structure  $\xi$  and let  $L$  be a closed connected Legendrian submanifold  $L$  of  $\Sigma$ . (If  $k = 1$ , the contact structure  $\xi$  is formed by the zero subspaces of the tangent spaces of  $\Sigma$  and the Legendrian submanifold  $L$  is just a point).

Let us make the following additional choices:

(C1) Pick a contact 1-form  $\lambda_0$  on  $\Sigma$ ,  $\xi = \ker \lambda_0$  (if  $k = 1$  we let  $\lambda_0$  be any non-vanishing 1-form on  $\Sigma$ ). Denote by  $\psi_t : \Sigma \rightarrow \Sigma$  the Reeb flow of  $\lambda_0$ . (Recall that a contact form  $\lambda_0$  defines a vector field, called Reeb vector field  $v$ , by two conditions:  $i_v d\lambda_0 = 0$  and  $\lambda_0(v) = 1$ . The flow of  $v$  is called the Reeb flow of  $\lambda_0$ ).

(C2) Pick any  $T > 0$  such that  $\psi_t(L)$  is well-defined and disjoint from  $L$  for all  $t \in (0, T]$  – such a  $T$  exists since  $L$  closed and tangent to the contact structure  $\xi$  while the Reeb vector field  $v$  generating the flow  $\{\psi_t\}$  is nowhere tangent to  $\xi$ .

(C3) Let  $0 < R_0 < R_1$ .

(C4) Let  $K$  be a closed connected manifold identified with the zero-section of  $T^*K$ .

We will view the symplectization of  $(\Sigma, \xi)$  as the symplectic manifold  $(\Sigma \times \mathbb{R}_+, d(s\lambda_0))$ , where  $s$  is the coordinate on the  $\mathbb{R}_+$ -factor of  $\Sigma \times \mathbb{R}_+$ . Consider the following quadruple of Lagrangian submanifolds (with boundary)  $\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H} \subset \Sigma \times \mathbb{R}_+ \times T^*K$ :

$$\begin{aligned} \mathcal{F} &:= \bigcup_{0 \leq t \leq T} \psi_t(L) \times R_0 \times K, \quad \mathcal{C} := \bigcup_{0 \leq t \leq T} \psi_t(L) \times R_1 \times K, \\ \mathcal{L} &:= \psi_T(L) \times [R_0, R_1] \times K, \quad \mathcal{H} := L \times [R_0, R_1] \times K. \end{aligned}$$

**Definition 1.2.** The union  $\Lambda := \mathcal{F} \cup \mathcal{C} \cup \mathcal{L} \cup \mathcal{H}$  is called a *Lagrangian tetragon*. The sets  $\mathcal{F}$  and  $\mathcal{C}$  are called its *floor* and *ceiling* while  $\mathcal{L}$  and  $\mathcal{H}$  the *low* and the *high* walls, respectively.

We shall need the following glossary.

Assume that  $U \subset \Sigma$  is a domain,  $\mathcal{I} \subset \mathbb{R}_+$  is an open interval containing  $[R_0, R_1]$  and  $W$  is an open tubular neighborhood of  $K$  in  $T^*K$ , so that  $U \times \mathcal{I} \times W$  contains the Lagrangian tetragon  $\Lambda$ . The image of  $\Lambda$  in an arbitrary symplectic manifold  $(M, \omega)$  under a symplectic embedding  $U \times \mathcal{I} \times W \rightarrow M$  will be called a *Lagrangian tetragon in  $M$*  or a *transplant* of the *original* Lagrangian tetragon  $\Lambda \subset \Sigma \times \mathbb{R}_+ \times T^*K$  to  $(M, \omega)$ . For the remainder of this section we will use the same notation for an original Lagrangian tetragon and for its transplant.

A Lagrangian tetragon  $\Lambda = \mathcal{F} \cup \mathcal{C} \cup \mathcal{L} \cup \mathcal{H} \subset (M, \omega)$  is called (*stably/autonomously*)  $\kappa$ -*interlinked* if the pair  $(\mathcal{L}, \mathcal{H})$  (*stably/autonomously*)  $\kappa$ -interlinks the pair  $(\mathcal{F}, \mathcal{C})$  in  $(M, \omega)$ . Roughly speaking, every complete (time-periodic) Hamiltonian on  $(M, \omega)$  which is larger on the high wall than on the low wall admits a chord from the floor to the ceiling.

**Example 1.3** (Prototype example). Take  $\Sigma$  to be the circle  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$  equipped with the trivial 0-dimensional contact structure determined by the contact form  $\lambda_0 = du$ , where  $u$  is the coordinate on  $\mathbb{R}$ . Fix a point  $v \in \mathbb{S}^1$  considered as a Legendrian submanifold of  $\Sigma$ , along with some numbers  $0 < R_0 < R_1$  and  $0 < T < 1$ . Let  $K = \{\text{a point}\}$ . The corresponding

Lagrangian tetragon  $\Lambda$  in the symplectization of  $\Sigma$  is a quadrilateral of area  $a = T(R_1 - R_0)$  in the cylinder  $\mathbb{S}^1(u) \times \mathbb{R}_+(s)$ , with the area form  $ds \wedge du$ . It was shown in [4] that  $\Lambda$  is stably  $a$ -interlinked. Moreover, this result is sharp:  $\Lambda$  is not  $a'$ -interlinked with any  $a' < a$  (see Remark 5.5).

This example can be extended to higher dimensions in two different ways.

**Example 1.4.** Let  $T^*\mathbb{T}^k = \mathbb{R}^k \times \mathbb{T}^k$ ,  $k > 1$ , be the cotangent bundle of the torus with the canonical coordinates  $p \in \mathbb{R}^k$  and  $q \in \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$  so that the symplectic structure is given by  $dp \wedge dq$ . Let  $|\cdot|$  denote the Euclidean norm. Consider the contact manifold  $\Sigma = \{|p| = 1\} \subset T^*\mathbb{T}^k$  (called the space of co-oriented contact elements to the torus), equipped with the contact form  $\lambda_0 = pdq$ . The corresponding Reeb flow is simply the Euclidean geodesic flow  $(p, q) \rightarrow (p, q + pt)$ . The symplectization  $\Sigma \times \mathbb{R}_+$  is identified symplectically with  $T^*\mathbb{T}^k \setminus \mathbb{T}^k$  by  $(p, q, s) \mapsto (sp, q)$ . Consider the Lagrangian tetragon  $\Lambda$  in  $T^*\mathbb{T}^k$  associated to the Legendrian fiber  $\{|p| = 1, q = 0\}$  of  $\Sigma$ , some  $0 < R_0 < R_1$ ,  $T > 0$  and  $K = \{\text{a point}\}$ . It is well-defined provided  $0 < T < 1/2$ . Observe that its floor, ceiling and the low wall lie in the boundary of the domain

$$E := \{R_0 < |p| < R_1\} \times \{|q| < T\},$$

while the high wall is the central fiber of the fibration  $E \rightarrow \{|q| < T\}$ .

**Theorem 1.5.**  $\Lambda$  is stably  $\kappa$ -interlinked with  $\kappa = T(R_1 - R_0)$ .

For the proof see Section 5.2.

**Example 1.6.** Consider the unit sphere  $\mathbb{S}^{2k-1}$  in the standard symplectic space  $\mathbb{R}^{2k} = \mathbb{C}^k$  equipped with the complex coordinates  $z = p + iq$  (in the vector notation) and the symplectic form  $dp \wedge dq$ . The sphere carries a contact structure  $\xi_{st}$  given by its field of complex tangencies which is defined by the contact form  $\lambda_0 = \frac{1}{2}(pdq - qdp)$ . The corresponding Reeb flow is given by  $z \mapsto e^{2it}z$ . The symplectization  $\Sigma \times \mathbb{R}_+$  is identified symplectically with  $\mathbb{C}^k \setminus 0$  by  $(z, s) \mapsto \sqrt{s}z$ . Consider the Lagrangian tetragon  $\Lambda$  in  $\mathbb{C}^k$  associated to the Legendrian sphere  $\{|p| = 1, q = 0\}$ , some numbers  $0 < R_0 < R_1$ ,  $T = \pi/4$ , and  $K = \{\text{a point}\}$ . Observe that the high and the low walls of  $\Lambda$  lie in spherical shells  $\{R_0^{1/2} \leq |p| \leq R_1^{1/2}, q = 0\}$  and  $\{p = 0, R_0^{1/2} \leq |q| \leq R_1^{1/2}\}$  and in the  $p$ - and in the  $q$ -space, respectively. Its floor and ceiling lie in the  $(2k - 1)$ -dimensional spheres in  $\mathbb{C}^k$  of radii  $R_0^{1/2}$  and  $R_1^{1/2}$ , respectively.

**Theorem 1.7.**  $\Lambda$  is stably  $\kappa$ -interlinked with  $\kappa = \pi(R_1 - R_0)/4$ .

For the proof see Section 5.3.

### 1.3 Superconductivity channels

Let us now present applications of the results above to specific Hamiltonian systems.

The first application is related to an example that appears in the famous work of Nekhoroshev [22] on the long-term stability of nearly integrable systems<sup>1</sup>.

Namely, let  $\phi, I$  denote the standard action-angle coordinates on the cotangent bundle  $T^*\mathbb{T}^k = \mathbb{T}^k \times \mathbb{R}^k$  of a torus  $\mathbb{T}^k$  (the  $\phi$  coordinates are defined mod 1 and the standard symplectic form on  $T^*\mathbb{T}^k$  is written as  $dI \wedge d\phi$ ). Nekhoroshev considered (analytic) Hamiltonians of the form

$$H(\phi, I) = h(I) + \epsilon f(I, \phi), \quad (2)$$

where  $\epsilon$  is a small parameter, and studied the long-term behavior  $I(t)$  of the action variables  $I$  along trajectories of the flow generated by  $H$ . His main discovery was that if  $h$  is a so-called steep function (that is, its restriction to any affine subspace of  $\mathbb{R}^k$  has only isolated critical points), then, as long as  $\epsilon$  is sufficiently small,  $I(t)$  stays close to  $I(0)$  for exponentially long times. At the same time Nekhoroshev gave an explicit example of a Hamiltonian system of type (2) with two degrees of freedom showing that if  $h$  is not steep, then even for small  $\epsilon$  the actions  $I(t)$  may grow linearly fast with  $t$  (the so-called “fast diffusion” phenomenon) along certain chords of  $H$  projecting to straight intervals in a level set of  $h$  in the  $I$ -coordinate space (the so-called “superconductivity channels”).

Here we present a multi-dimensional Nekhoroshev-type example where the linear growth phenomenon is robust with respect to perturbations of the Hamiltonian that are  $C^0$ -small on a certain “thin” subset of the phase space (but may be  $C^1$ -large everywhere).

Namely, consider the time-independent Hamiltonian

$$H(p, q, p', q') = \sum_{i=1}^k p_i h_i(p') + U(q) \quad (3)$$

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<sup>1</sup>Our interest to this model was triggered by a recent paper [6] by Bounemoura and Kaloshin.

on the symplectic manifold

$$M := T^*\mathbb{T}^k(p, q) \times T^*\mathbb{T}^m(p', q') .$$

Here the  $p, p'$  and  $q, q'$ -coordinates correspond, respectively, to the  $I$  and  $\phi$ -coordinates above,  $h_i$  are arbitrary smooth functions with  $h_i(0) = 0$ , and  $U$  is a non-constant potential. One easily checks that  $H$  is complete. If, for instance, all  $h_i$ 's are linear,  $H$  is quadratic, albeit non-convex, in momenta.

Looking at the Hamiltonian flow of  $H$  one readily sees that if  $p'(0) = 0$ , then  $q(t) = q(0)$  for all  $t$  and  $p(t) = p(0) - \nabla U(q(0)) \cdot t$ . Thus choosing  $q(0)$  to be a non-critical point of the potential, we see that the increment of the momentum  $|p(t) - p(0)|$  grows with linear speed along a straight line (a *superconductivity channel*).

Assume now that the potential  $U$  attains a local maximum  $\beta := U(0)$  at the point  $0 \in \mathbb{T}^k$ . Take  $r < 1/2$  and assume that  $\alpha := \max_{\{|q|=r\}} U < \beta$ .

Take the Lagrangian tetragon in  $T^*\mathbb{T}^k$  as in Example 1.3, if  $k = 1$ , and as in Example 1.4, if  $k > 1$ . Stabilizing it by the zero-section  $\mathbb{T}^m \subset T^*\mathbb{T}^m$ , we get a Lagrangian tetragon in  $M = T^*\mathbb{T}^k \times T^*\mathbb{T}^m$ . Denote by  $\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}$  its floor, ceiling and walls. Observe that  $H$   $\gamma$ -separates the walls  $\mathcal{L}$  and  $\mathcal{H}$  with  $\gamma := \beta - \alpha$ . As an immediate consequence of Theorem 1.5 (if  $k > 1$ ) and of the statement at the end of Example 1.3 (if  $k = 1$ ) we get the following result.

**Corollary 1.8.** *For every complete Hamiltonian  $G = H + F$ , where  $F \in C^\infty(M \times \mathbb{S}^1)$  and  $\delta := |\Delta(F; \mathcal{L}, \mathcal{H})| < \gamma$ , there exists a Hamiltonian chord of  $G$  from  $\mathcal{F}$  to  $\mathcal{C}$  of time-length  $\leq (R_1 - R_0)r/(\gamma - \delta)$ . The increment of  $|p|$  along this chord equals  $R_1 - R_0$ .*

## 1.4 Unstable equilibrium

The simplest model of an unstable equilibrium is provided by the quadratic Hamiltonian  $H = \frac{1}{2}(|p|^2 - |q|^2)$  on the standard symplectic vector space  $(\mathbb{R}^{2k}, dp \wedge dq)$  (as before,  $|\cdot|$  denotes the Euclidean norm). Consider the Lagrangian tetragon  $\Lambda \subset \mathbb{R}^{2k}$  described in Example 1.6. Denote by  $\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}$  its floor, ceiling and walls. Take any point of the form  $z = u$ ,  $u \in \mathbb{R}^k$ , lying in the sphere  $\{|p| = R_0^{1/2}, q = 0\} \subset \mathcal{H}$ . By definition, the point  $z' = e^{i\pi/4}z$  lies on the floor  $\mathcal{F}$ . At the same time one readily sees that  $z'$  belongs to the unstable manifold  $\{p = q\}$  of the fixed point 0 of the Hamiltonian flow of  $H$ .



The trajectory of  $z'$  has the form  $e^t z'$ , and hence it eventually hits the ceiling  $\mathcal{C}$ . Thus we have found a chord of  $H$  from  $\mathcal{F}$  to  $\mathcal{C}$ .

Observe also that  $H$   $\gamma$ -separates  $\mathcal{L}$  and  $\mathcal{H}$  with  $\gamma = R_0$ . Therefore, by Theorem 1.7, a chord connecting  $\mathcal{F}$  and  $\mathcal{C}$  persists under sufficiently  $C^0$ -small perturbations of  $H$  on  $\mathcal{L}$  and  $\mathcal{H}$ , yielding the following corollary.

**Corollary 1.9.** *For every complete Hamiltonian  $G = H + F$ , where  $F \in C^\infty(M \times \mathbb{S}^1)$  and  $\delta := |\Delta(F; \mathcal{L}, \mathcal{H})| < R_0$ , there is a Hamiltonian chord of  $G$  from  $\mathcal{F}$  to  $\mathcal{C}$  of time-length  $\leq \pi(R_1 - R_0)/(4(R_0 - \delta))$ . The increment of  $|(p, q)|$  along this chord equals  $R_1^{1/2} - R_0^{1/2}$ .*

Here is another similar setting where Theorem 1.7 can be applied. Let  $G : \mathbb{R}^{2k}(p, q) \times \mathbb{S}^1(t) \rightarrow \mathbb{R}$  be a complete Hamiltonian of the form

$$G(p, q, t) = |p|^2/2 + U(q, t).$$

Assume that for some  $0 < R_0 < R_1$

$$\max_{\{R_0^{1/2} \leq |q| \leq R_1^{1/2}\} \times \mathbb{S}^1} U =: -\beta \leq 0$$

and that  $U(0, t) = 0$  for all  $t \in \mathbb{S}^1$ . Then  $G$   $\gamma$ -separates  $\mathcal{L}$  and  $\mathcal{H}$  with  $\gamma = R_0/2 + \beta$ . Thus, Theorem 1.7 yields the following corollary.

**Corollary 1.10.** *There exists a Hamiltonian chord of  $G$  from  $\mathcal{F}$  to  $\mathcal{C}$  of time-length bounded from above by  $\pi(R_1 - R_0)/(2R_0 + 4\beta)$ .*

This corollary can be viewed as a generalization of the known result in the theory of the inverse Lagrange-Dirichlet problem about the instability of the equilibrium point of a mechanical system corresponding to a (non-strict) local maximum of the potential (see [16, 23]) – such an instability follows immediately if the potential  $U$  above is taken to be time-independent and with a local maximum  $U(0) = 0$  at  $q = 0$ .

## 1.5 A Poisson bracket invariant

It has been shown in [4] that the existence of connecting trajectories of Hamiltonian flows can be proved by means of a certain invariant involving the Poisson bracket. In the present paper we refine this techniques in order to establish interlinking for Lagrangian tetragons.

Given a symplectic manifold  $(M, \omega)$  and a quadruple of compact sets  $X_0, X_1, Y_0, Y_1 \subset M$  with  $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$ , define a number

$$pb_4^+(X_0, X_1, Y_0, Y_1) := \inf \max_M \{F, G\} ,$$

where the infimum is taken over all compactly supported functions  $F, G : M \rightarrow \mathbb{R}$  such that  $F|_{X_0} \leq 0$ ,  $F|_{X_1} \geq 1$ ,  $G|_{Y_0} \leq 0$ ,  $G|_{Y_1} \geq 1$ . Let us explain the notation:  $pb$  stands for the ‘‘Poisson bracket’’, 4 for the number of sets and  $+$  for the fact that we are considering the maximum of the Poisson bracket, instead of the uniform norm as it was done in [4].

Write  $\widehat{X}_i, \widehat{Y}_i$  for the  $\mathbb{S}^1$ -stabilizations of  $X_i$  and  $Y_i$  in  $M \times T^*\mathbb{S}^1$ .

Consider the set  $\Upsilon$  (resp.,  $\Upsilon_{aut}$ ) of all  $\kappa > 0$  such that the pair  $(Y_0, Y_1)$   $\kappa$ -interlinks (resp., autonomously  $\kappa$ -interlinks) the pair  $(X_0, X_1)$ . Let

$$\bar{\kappa} := \inf \Upsilon, \quad \bar{\kappa}_{aut} := \inf \Upsilon_{aut}.$$

If  $\Upsilon = \emptyset$  (resp.,  $\Upsilon_{aut} = \emptyset$ ), set  $\bar{\kappa} := +\infty$  (resp.,  $\bar{\kappa}_{aut} := +\infty$ ). One easily checks that  $\bar{\kappa}_{aut} \leq \bar{\kappa}$  and  $\Upsilon = [\bar{\kappa}, +\infty) \subset \Upsilon_{aut} = [\bar{\kappa}_{aut}, +\infty)$ .

**Theorem 1.11.**

$$1/pb_4^+(X_0, X_1, Y_0, Y_1) = \bar{\kappa}_{aut} \leq \bar{\kappa} \leq 1/pb_4^+(\widehat{X}_0, \widehat{X}_1, \widehat{Y}_0, \widehat{Y}_1).$$

*In particular, if  $pb_4^+(\widehat{X}_0, \widehat{X}_1, \widehat{Y}_0, \widehat{Y}_1) =: 1/\kappa > 0$ , then the pair  $(Y_0, Y_1)$   $\kappa$ -interlinks the pair  $(X_0, X_1)$ , and if  $pb_4^+(X_0, X_1, Y_0, Y_1) =: 1/\kappa > 0$ , then the pair  $(Y_0, Y_1)$  autonomously  $\kappa$ -interlinks the pair  $(X_0, X_1)$ .*

The proof follows the lines of [4] with the following amendments.

1. We adjust the argument of [4] to complete but not necessarily compactly supported Hamiltonians appearing in the definition of interlinking.
2. The results of [4] do not say whether the Hamiltonian chord connecting  $X_0$  and  $X_1$  goes from  $X_0$  to  $X_1$  or from  $X_1$  to  $X_0$ . This is why we introduce a refined version of the Poisson bracket invariant and use a recent theorem of A.Fathi (see Theorem 3.1) to detect Hamiltonian chords going between two sets in a given direction.

## 1.6 Interlinking and weakly exact Lagrangians

Let us discuss our method of proof of the interlinking of a Lagrangian tetragon. For the sake of transparency, let us focus on the simplest case of autonomous interlinking of a Lagrangian tetragon  $\Lambda = \mathcal{F} \cup \mathcal{C} \cup \mathcal{L} \cup \mathcal{H}$  in the symplectization  $\Sigma \times \mathbb{R}_+$  of a contact manifold  $(\Sigma, \xi)$ . By Theorem 1.11, in order to establish the autonomous interlinking for a Lagrangian tetragon  $\Lambda$ , it suffices to show the positivity of  $pb_4^+(\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H})$ . We will prove this positivity as follows.

Assume that  $\Lambda$  was constructed using a Legendrian submanifold  $L \subset (\Sigma, \xi)$ , a contact form  $\lambda_0$  on  $\Sigma$  and real parameters  $0 < R_0 < R_1$  and  $T > 0$ , see (C1)-(C3) in Section 1.2 above<sup>2</sup>.

The Lagrangian tetragon  $\Lambda$  is a singular Lagrangian submanifold with corners. One can smoothen its corners and get a smooth Lagrangian submanifold  $\Lambda_\varepsilon$  in  $\Sigma \times \mathbb{R}_+$  diffeomorphic to  $L \times \mathbb{S}^1$ . One easily checks that the Lagrangian isotopy class of  $\Lambda_\varepsilon$  depends only on the pair  $(\Sigma, L)$  and not on  $\lambda_0, R_0, R_1, T > 0$  and  $\varepsilon$ .

Recall that a closed Lagrangian submanifold of a symplectic manifold  $(M, \omega)$  is called *weakly exact* if the integral of  $\omega$  over any 2-disk in  $M$  with the boundary in the Lagrangian submanifold is zero.

We will say that the pair  $(\Sigma, L)$  is *weakly exact*, if the above-mentioned Lagrangian isotopy class of  $\Lambda_\varepsilon$  contains a weakly exact Lagrangian submanifold. Otherwise  $(\Sigma, L)$  will be called *weakly non-exact*.

**Theorem 1.12.** *Let  $(\Sigma, L)$  be a weakly non-exact pair and let  $\Lambda = \mathcal{F} \cup \mathcal{C} \cup \mathcal{L} \cup \mathcal{H} \subset \Sigma \times \mathbb{R}_+$  be a Lagrangian tetragon constructed using  $L$ , a contact form on  $(\Sigma, \xi)$  and some parameters  $R_0, R_1, T$ . Then*

$$pb_4^+(\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}) \geq \left( (R_1 - R_0)T \right)^{-1}.$$

and thus, by Theorem 1.11,  $\Lambda$  is autonomously  $(R_1 - R_0)T$ -interlinked.

We refer to Section 5.1 for the proof and a more general version of this result. Theorem 1.12 gives rise to the following question.

**Question 1.13.** *Do weakly exact pairs exist?*

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<sup>2</sup>The manifold  $K$  in (C4) is assumed to be the point.

Note that there do exist contact manifolds whose symplectizations contain *some* weakly exact closed Lagrangian submanifolds (see [21]) but we do not know whether such a weakly exact closed Lagrangian submanifold can be constructed as a smoothened Lagrangian tetragon.

Below (see Propositions 5.3, 5.9, 5.10) we describe three (partially overlapping) classes of *weakly non-exact* pairs  $(\Sigma, L)$ :

1.  $(\Sigma, \xi)$  is the unit cotangent bundle of a smooth manifold  $V$ ,  $L \subset \Sigma$  is any closed connected Legendrian submanifold such that the image of the homomorphism  $\pi_1(L) \rightarrow \pi_1(V)$ , induced by the projection  $L \rightarrow V$ , is finite. The proof relies on a theorem of Kragh [17].
2.  $(\Sigma, \xi)$  admits a symplectically aspherical strong symplectic semi-filling  $M$  (see Section 5.3 for the definition) and  $L \subset \Sigma$  is any closed connected Legendrian submanifold such that the homomorphism  $\pi_1(L) \rightarrow \pi_1(\Sigma)$  is trivial.
3.  $(\Sigma, \xi)$  admits a strong symplectic semi-filling  $M$  which is a subcritical Weinstein domain (see Section 5.3 for the definition) so that  $\pi_1(\Sigma) \rightarrow \pi_1(M)$  is injective, and  $L \subset \Sigma$  is any closed connected Legendrian submanifold. (In particular, this is true when  $(\Sigma, \xi)$  is the standard contact sphere). In fact, in such a case the symplectization of  $(\Sigma, \xi)$  does not admit any closed weakly exact Lagrangian submanifolds at all.

In the cases 2 and 3 weak non-exactness can be obtained by methods going back to the pioneering work of Gromov [15] (also see [3]).

**Remark 1.14.** Incidentally, Theorem 1.12 yields the following result in Reeb dynamics. Let  $(\Sigma, L)$  be a weakly non-exact pair and let  $\lambda_0$  be a contact form on  $(\Sigma, \xi)$  defining a Reeb flow  $\psi_t : \Sigma \rightarrow \Sigma$ , so that, as in the assumption (C2) in the construction of a Lagrangian tetragon,  $\psi_t(L)$  is well-defined and disjoint from  $L$  for all  $t \in (0, T]$  for some  $T > 0$ . Put  $L' := \psi_T(L)$ . Let  $\lambda$  be an arbitrary contact form on  $\Sigma$  that defines the same co-orientation of  $\xi$  as  $\lambda_0$  and has a complete Reeb flow.

We claim that the Reeb flow of  $\lambda$  has a chord from  $L$  to  $L'$ . Moreover, the time-length of this chord does not exceed  $T/C$ , where  $C := \min_Y (\lambda/\lambda_0)$  with  $Y := \bigcup_{t \in [0, T]} \psi_t(L)$ .

Indeed, look at the Lagrangian tetragon  $\Lambda = \mathcal{F} \cup \mathcal{C} \cup \mathcal{L} \cup \mathcal{H}$  associated to  $L$ ,  $\lambda_0$ ,  $T$  and parameters  $R =: R_1 > 1 =: R_0 > 0$ . Combining the anti-symmetry of the Poisson bracket invariant (see (6) below) with Theorem 1.12

we get that

$$pb_4^+(\mathcal{H}, \mathcal{L}, \mathcal{F}, \mathcal{C}) = ((R-1)T)^{-1},$$

and hence the pair  $(\mathcal{F}, \mathcal{C})$  autonomously  $\kappa$ -interlinks the pair of the walls  $(\mathcal{H}, \mathcal{L})$  with  $\kappa := (R-1)T$ . Choose  $R$  large enough and note that the contact Hamiltonian of  $\lambda$   $\Delta$ -separates  $\mathcal{F}$  and  $\mathcal{C}$  with  $\Delta := R \min_Y(\lambda/\lambda_0) - \max_Y(\lambda/\lambda_0)$ . Hence, for such  $R$ , the Reeb flow of  $\lambda$  admits a chord of time-length  $\leq \kappa/\Delta$  from  $L$  to  $L'$ . Letting  $R \rightarrow +\infty$ , we get the claim. Let us mention that Legendrian contact homology should be a more adequate and powerful technique for detecting Reeb chords connecting  $L$  and  $L'$ , see e.g. [10, 1].

**ORGANIZATION OF THE PAPER:** In Section 2 we recall the necessary preliminaries. In Section 3 we formulate a recent result of A.Fathi on the existence of chords for smooth vector fields which is crucial for our studies. In Section 4 we discuss the Poisson bracket invariants and their relation to the existence of Hamiltonian chords. Section 5 is the central section of the paper – in this section we prove the interlinking of Lagrangian tetragons.

## 2 Preliminaries

Let  $(M^{2n}, \omega)$  be a connected (not necessarily closed) symplectic manifold.

Given an open set  $U \subset M$ , set  $C_c^\infty(U)$  to be the space of compactly supported smooth functions on  $M$ .

Further on we always identify  $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ . Given a Hamiltonian  $G : M \times \mathbb{S}^1 = M \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ , we denote  $G_t := G(\cdot, t)$  and say that  $G$  is *compactly supported*, if  $\text{supp } G := \bigcup_{t \in \mathbb{S}^1} \text{supp } G_t$  is a compact subset of  $M$ . For a bounded  $G \in C^\infty(M)$  denote

$$||G|| := \sup_M |G|.$$

For  $G \in C^\infty(M)$  define a vector field  $\text{sgrad } G$  by  $i_{\text{sgrad } G} \omega = -dG$ . Given  $F, G \in C^\infty(M)$ , define the Poisson bracket  $\{F, G\}$  by

$$\begin{aligned} \{F, G\} &:= \omega(\text{sgrad } G, \text{sgrad } F) = dF(\text{sgrad } G) = -dG(\text{sgrad } F) = \\ &= L_{\text{sgrad } G} F = -L_{\text{sgrad } F} G. \end{aligned}$$

The *Hamiltonian flow*, or just a *flow*, of  $G : M \times \mathbb{S}^1 \rightarrow \mathbb{R}$  is, by definition, the flow of the (time-dependent) vector field  $\text{sgrad } G_t$ .

A subset of  $(M, \omega)$  is called *displaceable*, if it can be completely displaced from its closure by the flow of a compactly supported (time-dependent) Hamiltonian.

We say that a (possibly open) symplectic manifold is of *bounded geometry at infinity*, if it is geometrically bounded in the sense of [3] or convex at infinity in the sense of [11]. Let us note that for any smooth manifold  $N$  the cotangent bundle  $T^*N$ , equipped with the standard symplectic structure, is of bounded geometry at infinity.

For a topological space  $X$  denote by  $H_2^S(X) \subset H_2(X; \mathbb{R})$  the image of the Hurewicz homomorphism  $\pi_2(X) \rightarrow H_2(X; \mathbb{R})$ . If  $Y$  is a subset of  $X$ , we denote by  $H_2^D(X, Y) \subset H_2(X, Y; \mathbb{R})$  the image of the relative Hurewicz homomorphism  $\pi_2(X, Y) \rightarrow H_2(X, Y; \mathbb{R})$ .

A symplectic manifold  $(M, \omega)$  is called *exact*, if  $\omega$  is exact, and *symplectically aspherical* (sometimes it is also called *weakly exact*), if  $\omega(H_2^S(M; \mathbb{R})) = 0$ . An exact symplectic manifold is, of course, symplectically aspherical but not necessarily vice versa.

A Lagrangian submanifold  $L$  of a symplectic manifold  $(M, \omega)$  is called *weakly exact* if  $\omega(H_2^D(M, L; \mathbb{R})) = 0$ . A Lagrangian submanifold  $L$  of an exact symplectic manifold  $(M, d\lambda)$  is called *exact* if the cohomology class  $[\lambda|_L] \in H^1(L; \mathbb{R})$  is zero. An exact Lagrangian submanifold is weakly exact but not necessarily vice versa.

### 3 Connecting trajectories of smooth vector fields

In this section let  $M$  be any smooth manifold,  $v$  a complete smooth time-independent vector field on  $M$  and  $X_0, X_1 \subset M$  disjoint compact subsets of  $M$ . Denote by  $T(X_0, X_1; v)$  the infimum of the time-lengths of the chords of  $v$  from  $X_0$  to  $X_1$  (if there is no such chord, set  $T(X_0, X_1; v) := +\infty$ ). Define

$$L_{\max}(X_0, X_1; v) := \inf_F \max_M L_v F,$$

where the infimum is taken over all smooth compactly supported functions  $F$  on  $M$  such that  $F|_{X_0} \leq 0$ ,  $F|_{X_1} \geq 1$ .

A basic calculus argument shows that

$$T(X_0, X_1; v) \geq 1/L_{\max}(X_0, X_1; v).$$

The following theorem has been proved by A.Fathi.

**Theorem 3.1** (A.Fathi, [14]).  $T(X_0, X_1; v) = 1/L_{\max}(X_0, X_1; v)$ .

*In particular, if  $L_{\max}(X_0, X_1; v) > 0$ , then for any  $p > L_{\max}(X_0, X_1; v)$  there exists a chord of  $v$  from  $X_0$  to  $X_1$  of time-length  $\leq 1/p$ .*

**Remark 3.2.** Replace the maximum of  $L_v F$  by  $\|L_v F\|$  in the definition of  $L_{\max}(X_0, X_1; v)$  and call the resulting quantity  $L_0(X_0, X_1; v)$ . Then, as it was shown in [4, Section 4.1] by a rather basic averaging argument,

$$\min\{T(X_0, X_1; v), T(X_1, X_0; v)\} = 1/L_0(X_0, X_1; v). \quad (4)$$

In other words, if  $L_0(X_0, X_1; v) > 0$ , then there exists *either* a chord of  $v$  from  $X_0$  to  $X_1$  *or* a chord of  $v$  from  $X_1$  to  $X_0$ . (Note that none of the results in [4] about the existence of chords says anything about the direction of chords!)

Theorem 3.1 is much more difficult than (4) – in fact, Fathi’s proof of Theorem 3.1 uses, along with the averaging argument similar to the one in [4, Section 4.1], some ingenious arguments from the general theory of metric spaces.

## 4 Poisson bracket invariants

Let  $(M, \omega)$  be a connected symplectic manifold.

We say that sets  $X_0, X_1, Y_0, Y_1 \subset M$  form an *admissible quadruple*, if they are compact and  $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$ .

Assume  $U$  is an open subset of  $M$  and  $X_0, X_1, Y_0, Y_1 \subset U$  is an admissible quadruple. Define

$$pb_4^U(X_0, X_1, Y_0, Y_1) := \inf_{F, G} \|\{F, G\}\|,$$

$$pb_4^{U,+}(X_0, X_1, Y_0, Y_1) := \inf_{F, G} \max_M \{F, G\},$$

$$pb_4^{U,-}(X_0, X_1, Y_0, Y_1) := \inf_{F, G} (-\min_M \{F, G\}),$$

where the infimum in all the cases is taken over all  $F, G \in C_c^\infty(U)$  such that

$$F|_{X_0} \leq 0, F|_{X_1} \geq 1, G|_{Y_0} \leq 0, G|_{Y_1} \geq 1. \quad (5)$$

One can prove similarly to [4] that this class of pairs  $(F, G)$  can be replaced, without changing the infimums, by a smaller class where the inequalities for  $F, G$  are replaced by the equalities on some open neighborhoods of the sets  $X_0, X_1, Y_0, Y_1$ .

Let us now define a stabilized version of  $pb_4^{U, \pm}$ . Identify the cotangent bundle  $T^*\mathbb{S}^1$  with the cylinder  $\mathbb{R} \times \mathbb{S}^1$  equipped with the coordinates  $r$  and  $\theta \pmod{1}$  and the standard symplectic structure  $dr \wedge d\theta$ . As above, write  $\widehat{X}_i, \widehat{Y}_i$  for the  $\mathbb{S}^1$ -stabilizations of  $X_i$  and  $Y_i$  in  $M \times T^*\mathbb{S}^1$ . Given an open set  $U \subset M$  and an admissible quadruple  $X_0, X_1, Y_0, Y_1 \subset U$ , set

$$\begin{aligned} \widehat{pb}_4^{U, +}(X_0, X_1, Y_0, Y_1) &:= pb_4^{U \times T^*\mathbb{S}^1, +}(\widehat{X}_0, \widehat{X}_1, \widehat{Y}_0, \widehat{Y}_1), \\ \widehat{pb}_4^{U, -}(X_0, X_1, Y_0, Y_1) &:= pb_4^{U \times T^*\mathbb{S}^1, -}(\widehat{X}_0, \widehat{X}_1, \widehat{Y}_0, \widehat{Y}_1). \end{aligned}$$

**If  $U = M$ , or it is clear from the context what  $X_0, X_1, Y_0, Y_1$  are meant, we will omit the corresponding indices and sets in the notation for  $pb_4, pb_4^\pm$  and  $\widehat{pb}_4, \widehat{pb}_4^\pm$ .**

The following properties of  $pb_4^\pm, \widehat{pb}_4^\pm$  follow easily from the definitions, similarly to the corresponding properties for  $pb_4$  (cf. [4]):

ANTI-SYMMETRY:

$$\begin{aligned} pb_4^{U, +}(X_0, X_1, Y_0, Y_1) &= pb_4^{U, -}(X_1, X_0, Y_0, Y_1) = pb_4^{U, -}(X_0, X_1, Y_1, Y_0) = \\ &= pb_4^{U, -}(Y_0, Y_1, X_0, X_1) = pb_4^{U, +}(Y_1, Y_0, X_0, X_1), \end{aligned} \quad (6)$$

$$pb_4 \geq \max\{pb_4^+, pb_4^-\}. \quad (7)$$

Similar claims hold also for  $\widehat{pb}_4^{U, \pm}$ .

BEHAVIOR UNDER PRODUCTS:

Suppose that  $M$  and  $N$  are connected symplectic manifolds. Equip  $M \times N$  with the product symplectic form. Let  $K \subset N$  be a compact subset. Then for every collection  $X_0, X_1, Y_0, Y_1$  of compact subsets of  $M$

$$pb_4^{M \times N, \pm}(X_0 \times K, X_1 \times K, Y_0 \times K, Y_1 \times K) \leq pb_4^{M, \pm}(X_0, X_1, Y_0, Y_1) \quad (8)$$



and a similar claim holds also for  $\widehat{pb}_4^\pm$ .

In particular, for any  $U \subset M$ ,  $X_0, X_1, Y_0, Y_1 \subset U$ ,

$$\widehat{pb}_4^{U,\pm}(X_0, X_1, Y_0, Y_1) \leq pb_4^{U,\pm}(X_0, X_1, Y_0, Y_1).$$

**MONOTONICITY:** Assume  $U \subset W$  are opens sets in  $M$  and  $X'_0, X'_1, Y'_0, Y'_1 \subset U \subset W$  is an admissible quadruple. Let  $X_0, X_1, Y_0, Y_1$  be another admissible quadruple such that  $X_0 \subset X'_0, X_1 \subset X'_1, Y_0 \subset Y'_0, Y_1 \subset Y'_1$ . Then

$$\widehat{pb}_4^{W,\pm}(X_0, X_1, Y_0, Y_1) \leq \widehat{pb}_4^{U,\pm}(X'_0, X'_1, Y'_0, Y'_1). \quad (9)$$

**SEMI-CONTINUITY:**

Suppose that a sequence  $X_0^{(j)}, X_1^{(j)}, Y_0^{(j)}, Y_1^{(j)}$ ,  $j \in \mathbb{N}$ , of ordered collections converges (in the sense of the Hausdorff distance between sets) to a collection  $X_0, X_1, Y_0, Y_1$ . Then

$$\limsup_{j \rightarrow +\infty} pb_4^\pm(X_0^{(j)}, X_1^{(j)}, Y_0^{(j)}, Y_1^{(j)}) \leq pb_4^\pm(X_0, X_1, Y_0, Y_1). \quad (10)$$

A similar inequality holds also for  $\widehat{pb}_4^\pm$ .

**Proof of Theorem 1.11.** Let us prove that

$$1/pb_4^+(X_0, X_1, Y_0, Y_1) = \bar{\kappa}_{aut}. \quad (11)$$

To prove that

$$1/pb_4^+(X_0, X_1, Y_0, Y_1) \leq \bar{\kappa}_{aut}. \quad (12)$$

we need to prove that

$$pb_4^+(X_0, X_1, Y_0, Y_1) \geq 1/\kappa \quad (13)$$

for any  $\kappa \in \Upsilon_{aut}$ . Pick such a  $\kappa \in \Upsilon_{aut}$  and any  $F, G \in C_c^\infty(M)$  satisfying (5). Then the pair  $(Y_0, Y_1)$  autonomously  $\kappa$ -interlinks the pair  $(X_0, X_1)$  and, since  $G$  1-separates  $Y_0$  and  $Y_1$ , this means that there exists a chord of  $G$  from  $X_0$  to  $X_1$  of time-length  $\leq \kappa$ . Restricting  $F$  to the chord and applying the mean value theorem from the basic calculus one readily obtains that the function  $L_{\text{sgrad } G}F = \{F, G\}$  takes a value greater or equal to  $1/\kappa$  at some point of the chord and thus  $\max_M \{F, G\} \geq 1/\kappa$ . Since this holds for any  $F, G \in C_c^\infty(M)$  satisfying (5), we obtain (13) and hence (12).

Let us prove that

$$k_{aut} := 1/pb_4^+(X_0, X_1, Y_0, Y_1) \leq \bar{\kappa}_{aut}. \quad (14)$$

Equivalently, this means to prove the following: Let  $G : M \rightarrow \mathbb{R}$  be a complete Hamiltonian that  $\Delta$ -separates  $Y_0$  and  $Y_1$ . We need to show that there exists a chord of  $G$  from  $X_0$  to  $X_1$  of time-length  $\leq \kappa_{aut}/\Delta$ .

We may assume without loss of generality that  $\max_{Y_0} G = 0$ ,  $\min_{Y_1} G = 1$ ,  $\Delta = 1$  (the general case is reduced to this one if one replaces  $G$  with  $u \circ G$  for an appropriate function  $u : \mathbb{R} \rightarrow \mathbb{R}$ ). Let  $g_t$  be the flow of  $G$ . Since the flow  $g_t$  is defined for all  $t \in \mathbb{R}$ , the following subset of  $M$  is well-defined and compact:

$$\Theta_{aut} := \bigcup_{0 \leq t \leq \kappa_{aut}} g_t(X_0) \cup Y_0 \cup Y_1.$$

Let  $G' : M \rightarrow \mathbb{R}$  be a compactly supported Hamiltonian which coincides with  $G$  on an open neighborhood of  $\Theta_{aut}$ . Since  $G$  coincides with  $G'$  on  $Y_0$  and  $Y_1$ , we get that

$$\max_{Y_0} G = \max_{Y_0} G' = 0, \quad \min_{Y_1} G = \min_{Y_1} G' = 1.$$

Since

$$\kappa_{aut} = 1/pb_4^+(X_0, X_1, Y_0, Y_1) > 0,$$

we get that

$$\begin{aligned} \max_M \{F', G'\} &= \max_M L_{\text{sgrad } G'} F' \geq \\ &\geq L_{\max}(X_0, X_1; \text{sgrad } G') \geq pb_4^+(X_0, X_1, Y_0, Y_1) = 1/\kappa_{aut} \end{aligned}$$

for any compactly supported  $F' : M \rightarrow \mathbb{R}$  such that  $F'|_{X_0} \leq 0$ ,  $F'|_{Y_1} \geq 1$ . Therefore, by Theorem 3.1, there exists a chord of  $G'$  from  $X_0$  to  $X_1$  of time-length  $\leq \kappa_{aut}$ . This chord of  $G'$  is also a chord of  $G$  – indeed,  $G'$  coincides with  $G$  on a neighborhood of  $\Theta_{aut}$  and therefore for any  $t \in [0, \kappa_{aut}]$  the time- $[0, t]$  flows of  $G$  and  $G'$  coincide on  $X_0$ .

Thus we have proved the existence of a chord of  $G$  from  $X_0$  to  $X_1$  of time-length  $\leq \kappa_{aut}$ . This finishes the proof of (14). Together (12) and (14) imply (11).

Let us now prove that

$$\bar{\kappa} \leq \kappa := 1/pb_4^+(\hat{X}_0, \hat{X}_1, \hat{Y}_0, \hat{Y}_1). \quad (15)$$

Equivalently, this means to prove the following: Let  $G : M \times \mathbb{S}^1 \rightarrow \mathbb{R}$  be a complete Hamiltonian that  $\Delta$ -separates  $Y_0$  and  $Y_1$ . We need to show that there exists a chord of  $G$  from  $X_0$  to  $X_1$  of time-length  $\leq \kappa/\Delta$ .

The argument is very similar to the argument used above in the autonomous case. Namely, consider an *autonomous* Hamiltonian

$$H : M \times T^*\mathbb{S}^1 \rightarrow \mathbb{R}, (x, r, \theta) \rightarrow G(x, \theta) + r,$$

generating a Hamiltonian flow  $h_t : M \times T^*\mathbb{S}^1 \rightarrow M \times T^*\mathbb{S}^1$ . Note that the projection  $M \times T^*\mathbb{S}^1 \rightarrow M$  maps each trajectory of  $h_t$  to a trajectory of  $g_t$  of the same time-length. Thus, it suffices to prove the existence of a chord of  $H$  from  $\widehat{X}_0$  to  $\widehat{X}_1$  of time-length  $\leq \kappa/\Delta$ .

The Hamiltonian flow  $h_t$  of  $H$  is defined for all times, since so is the Hamiltonian flow  $g_t$  of  $G$ . Note that  $r = 0$  on  $\widehat{Y}_0, \widehat{Y}_1$ . Therefore

$$\max_{\widehat{Y}_0} H = \max_{Y_0 \times \mathbb{S}^1} G, \quad \min_{\widehat{Y}_1} H = \min_{Y_1 \times \mathbb{S}^1} G,$$

meaning that  $H$   $\Delta$ -separates  $\widehat{Y}_0$  and  $\widehat{Y}_1$ . Thus,  $H$  is a complete autonomous Hamiltonian on  $M \times T^*\mathbb{S}^1$  that  $\Delta$ -separates  $\widehat{Y}_0$  and  $\widehat{Y}_1$ . By the result in the autonomous case,  $H$  has a chord from  $\widehat{X}_0$  to  $\widehat{X}_1$  of time-length  $\leq \kappa/\Delta$ , as required. This finishes the proof of (15) and of the theorem.  $\square$

## 5 Lagrangian tetragons

The proof of the positivity of  $\widehat{pb}_4$  for Lagrangian tetragons relies on the following proposition which is based on a method from [4] relating deformations of symplectic forms and Poisson brackets.

**Proposition 5.1.** *Let  $(M, \omega)$  be a connected (possibly open) symplectic manifold and let  $\Lambda \subset M$  be a closed Lagrangian submanifold. Let  $F, G \in C_c^\infty(M)$ . Assume*

1.  *$(M, \omega)$  does not admit weakly exact closed Lagrangian submanifolds Lagrangian isotopic to  $\Lambda$ .*
2. *There exists a number  $c > 0$  so that  $dF \wedge dG(A) = -c\omega(A)$  for any  $A \in H_2^D(M, \Lambda; \mathbb{R})$ .*
3. *The form  $dF \wedge dG$  vanishes on  $L$ .*

Then

$$\max_M \{F, G\} \geq c.$$

**Proof of Proposition 5.1.** Following [4], consider the deformation

$$\omega_\tau := \omega + \tau dF \wedge dG, \quad \tau \in \mathbb{R}.$$

A direct calculation shows that

$$dF \wedge dG \wedge \omega^{n-1} = -\frac{1}{n} \{F, G\} \omega^n.$$

and thus

$$\omega_\tau^n = (1 - \tau \{F, G\}) \omega^n.$$

Thus  $\omega_\tau$  is symplectic for any  $\tau \in I := [0, 1/\max_M \{F, G\})$ .

Fix an arbitrary  $\tau \in I$ . The form  $\omega$  can be mapped (using Moser's method [20]) to  $\omega_\tau$  by a compactly supported isotopy  $\vartheta_\tau : (M, \omega_\tau) \rightarrow (M, \omega)$ . Note that, by condition 3,  $L$  is Lagrangian with respect to  $\omega_\tau$ . Therefore the manifold  $\Lambda_\tau := \vartheta_\tau(\Lambda)$  is a Lagrangian submanifold of  $(M, \omega)$  Lagrangian isotopic to  $\Lambda$ .

The diffeomorphism  $\vartheta_\tau$  induces an isomorphism  $\vartheta_{\tau,*} : H_2^D(M, \Lambda; \mathbb{R}) \rightarrow H_2^D(M, \Lambda_\tau; \mathbb{R})$ . Then for any  $A \in H_2^D(M, \Lambda; \mathbb{R})$

$$\omega(\vartheta_{\tau,*}(A)) = \omega_\tau(A) = \omega(A) + \tau dF \wedge dG(A) = (1 - c\tau)\omega(A).$$

The above is true for any  $\tau \in I$  – thus, since we assumed that  $1/c \in I$ , it has to be true also for  $\tau = 1/c$ , meaning that  $\Lambda_{1/c}$  has to be weakly exact, contrary to the assumptions of the proposition. Thus,  $\tau = 1/c$  cannot lie inside  $I$ , meaning that  $1/c \geq 1/\max_M \{F, G\}$  or, equivalently, that  $\max_M \{F, G\} \geq c$ .  $\square$

## 5.1 Construction and interlinking of Lagrangian tetragons

Let  $(\Sigma^{2k-1}, \xi)$ ,  $k \geq 1$ , be a connected (not necessarily closed) contact manifold with a co-orientable contact structure  $\xi$  (if  $k = 1$ , the contact structure  $\xi$  is formed by the zero subspaces of the tangent spaces). Let  $L$

be a closed connected Legendrian submanifold of  $\Sigma$  (in the case  $k = 1$  the submanifold  $L$  is just a point).

Pick a contact form  $\lambda_0$  on  $\Sigma$ :  $\xi = \{\lambda_0 = 0\}$ . (In the case  $k = 1$  let  $\lambda_0$  be any non-vanishing form on  $\Sigma$ ). Denote by  $\psi_t : \Sigma \rightarrow \Sigma$  the Reeb flow of  $\lambda_0$ . (In the case  $k = 1$  the Reeb vector field  $v$  is defined just by the condition  $\lambda_0(v) \equiv 1$ ). Pick  $T > 0$  so that  $\psi_t(L)$  is well-defined and disjoint from  $L$  for all  $t \in (0, T]$ . Let  $0 < R_0 < R_1$ .

Let  $(\Sigma \times \mathbb{R}_+, d(s\lambda_0))$  be the symplectization of  $(\Sigma, \xi)$  (here  $s$  is the coordinate on the  $\mathbb{R}_+$ -factor of  $\Sigma \times \mathbb{R}_+$ ).

Define a *Lagrangian tetragon*  $\Lambda'' \subset \Sigma \times \mathbb{R}_+$ :

$$\begin{aligned}\mathcal{F}'' &:= \bigcup_{0 \leq t \leq T} \psi_t(L) \times R_0, \quad \mathcal{C}'' := \bigcup_{0 \leq t \leq T} \psi_t(L) \times R_1, \\ \mathcal{L}'' &:= \psi_T(L) \times [R_0, R_1], \quad \mathcal{H}'' := L \times [R_0, R_1], \\ \Lambda'' &:= \mathcal{F}'' \cup \mathcal{C}'' \cup \mathcal{L}'' \cup \mathcal{H}''.\end{aligned}$$

This is a singular Lagrangian submanifold. We will define its smoothening as follows.

Consider an embedding  $\Phi : L \times (\mathbb{R}_+ \times [0, T]) \rightarrow \Sigma \times \mathbb{R}_+$  given by

$$\Phi(x \times (s, t)) := (\psi_t(x), s).$$

Then for any smooth embedded loop  $\gamma \subset [R_0, R_1] \times [0, T]$  the restriction of  $\Phi$  to  $L \times \gamma$  is a Lagrangian embedding. Choose a family  $\gamma_\varepsilon$  of smooth embedded loops in  $[R_0, R_1] \times [0, T]$   $C^0$ -converging to the boundary of the rectangle  $[R_0, R_1] \times [0, T]$  as  $\varepsilon \rightarrow 0$  and denote by

$$\Lambda''_\varepsilon := \Phi(L \times \gamma_\varepsilon) \subset \Sigma \times \mathbb{R}_+$$

the resulting smooth Lagrangian submanifold, called a *smoothened Lagrangian tetragon* in  $\Sigma \times \mathbb{R}_+$ .

Let  $K$  be a closed connected manifold. Then

$$\Lambda' := \Lambda'' \times K$$

is a *Lagrangian tetragon* in  $\Sigma \times \mathbb{R}_+ \times T^*K$  and  $\Lambda'_\varepsilon := \Lambda''_\varepsilon \times K$  is its smoothening.

Assume now that  $U$  is a domain in  $\Sigma$ ,  $\mathcal{I} \subset \mathbb{R}_+$  is an open interval containing  $[R_0, R_1]$  and  $W$  is an open tubular neighborhood of  $K$  in  $T^*K$  so that  $\Lambda'$  is contained in  $U \times \mathcal{I} \times W$ . Let  $(M, \omega)$  be a connected symplectic

manifold and let  $\nu : U \times \mathcal{I} \times W \rightarrow M$  be a symplectic embedding. Denote by  $\Lambda := \nu(\Lambda')$  the resulting *Lagrangian tetragon* in  $M$  and let

$$\mathcal{F} := \nu(\mathcal{F}'' \times K), \mathcal{C} := \nu(\mathcal{C}'' \times K), \mathcal{L} := \nu(\mathcal{L}'' \times K), \mathcal{H} := \nu(\mathcal{H}'' \times K),$$

be, respectively, its floor, ceiling, low wall and high wall. Let  $\Lambda_\varepsilon := \nu(\Lambda'_\varepsilon)$  be its smoothening.

Note that for sufficiently small  $\varepsilon$  all Lagrangian submanifolds  $\Lambda_\varepsilon$  are Lagrangian isotopic in  $(M, \omega)$ . From this point on we consider only such  $\varepsilon$ .

Denote by

$$\nu_\# : H_2^D(U \times \mathcal{I} \times W, \Lambda'; \mathbb{R}) \rightarrow H_2^D(M, \Lambda; \mathbb{R})$$

the homomorphism induced by  $\nu$  and denote by

$$\partial : H_2^D(M, \Lambda; \mathbb{R}) \rightarrow H_1(\Lambda; \mathbb{R})$$

the boundary homomorphism. Note the images of the homology groups  $H_1(\mathcal{F}; \mathbb{R})$ ,  $H_1(\mathcal{C}; \mathbb{R})$ ,  $H_1(\mathcal{L}; \mathbb{R})$ ,  $H_1(\mathcal{H}; \mathbb{R})$  in  $H_1(\Lambda; \mathbb{R})$  under the homomorphisms induced by the inclusions all coincide (moreover, they are all isomorphic to  $H_1(L \times K; \mathbb{R})$ , since  $\nu$  is an embedding). Denote the resulting subgroup of  $H_1(\Lambda; \mathbb{R})$  by  $\Xi$ . There are natural isomorphisms

$$H_2^D(U \times \mathcal{I} \times W, \Lambda'; \mathbb{R}) \cong H_2^D(U \times \mathcal{I} \times W, \Lambda'_\varepsilon; \mathbb{R}),$$

$$H_2^D(M, \Lambda; \mathbb{R}) \cong H_2^D(M, \Lambda_\varepsilon; \mathbb{R}),$$

$$H_1(\Lambda; \mathbb{R}) \cong H_1(\Lambda_\varepsilon; \mathbb{R}).$$

We will use the same notation for the corresponding maps

$$\nu_\# : H_2^D(U \times \mathcal{I} \times W, \Lambda'_\varepsilon; \mathbb{R}) \rightarrow H_2^D(M, \Lambda_\varepsilon; \mathbb{R}),$$

$$\partial : H_2^D(M, \Lambda_\varepsilon; \mathbb{R}) \rightarrow H_1(\Lambda_\varepsilon; \mathbb{R})$$

and for the corresponding subgroup  $\Xi$  of  $H_1(\Lambda_\varepsilon; \mathbb{R})$ .

**Proposition 5.2.** *Assume that*

1.  $H_2^D(M, \Lambda; \mathbb{R}) = \text{Im } \nu_\# + Q$ , where  $Q \subset H_2^D(M, \Lambda; \mathbb{R})$  is a subgroup such that  $\omega(Q) = 0$  and  $\partial(Q) \subset \Xi$ .
2.  $M$  does not admit closed weakly exact Lagrangian submanifolds Lagrangian isotopic to  $\Lambda_\varepsilon$ .

Then

$$pb_4^{M,+}(\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}) \geq \left( (R_1 - R_0)T \right)^{-1} > 0,$$

and thus,  $\Lambda$  is autonomously  $(R_1 - R_0)T$ -interlinked.

The autonomous interlinking of  $\Lambda$  stated at the end of the theorem follows, by Theorem 1.11, from the lower bound on  $pb_4^{M,+}(\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H})$ .

**Proof of Theorem 1.12:** In the case  $K = \{\text{a point}\}$ ,  $M = \Sigma \times \mathbb{R}_+$  and  $\nu = Id$ , condition 1 of Proposition 5.2 is obviously satisfied and condition 2 just says that the pair  $(\Sigma, L)$  is weakly non-exact (see Section 1.6 for the definition of this notion). Thus, in this case Proposition 5.2 immediately proves Theorem 1.12.  $\square$

**Proof of Proposition 5.2.** The sets  $\Lambda_\varepsilon$   $C^0$ -converge to  $\Lambda$  as  $\varepsilon \rightarrow 0$ . Moreover, we can subdivide  $\Lambda_\varepsilon$  into four sets  $\mathcal{F}_\varepsilon, \mathcal{C}_\varepsilon, \mathcal{L}_\varepsilon, \mathcal{H}_\varepsilon$  that  $C^0$ -converge, respectively, to  $\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}$  as  $\varepsilon \rightarrow 0$ . In view of (10) we need to prove that

$$pb_4^{M,+}(\mathcal{F}_\varepsilon, \mathcal{C}_\varepsilon, \mathcal{L}_\varepsilon, \mathcal{H}_\varepsilon) \geq \left( (R_1 - R_0)T - \delta(\varepsilon) \right)^{-1}$$

for some  $\delta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Pick arbitrary smooth functions  $F, G \in C_c^\infty(M)$  satisfying the following conditions:

$$F|_{Op(\mathcal{F}_\varepsilon)} = G|_{Op(\mathcal{L}_\varepsilon)} = 0, \quad F|_{Op(\mathcal{C}_\varepsilon)} = G|_{Op(\mathcal{H}_\varepsilon)} = 1. \quad (16)$$

(Here  $Op$  stands for *some* open neighborhood of a set). Note that, by (16),

$$dF \wedge dG|_{Op(\Lambda_\varepsilon)} \equiv 0. \quad (17)$$

and thus  $FdG|_{\Lambda_\varepsilon}$  is a closed form.

Denote by  $D_{\gamma_\varepsilon}$  the disk bounded by  $\gamma_\varepsilon$  in  $[R_0, R_1] \times [0, T]$ . Set

$$\Gamma_\varepsilon := \Phi(\{\text{point}\} \times \gamma_\varepsilon) \subset \Lambda_\varepsilon''$$

and let

$$D_{\Gamma_\varepsilon} := \Phi(\{\text{point}\} \times D_{\gamma_\varepsilon})$$

be the disk bounded by  $\Gamma_\varepsilon$ . Note that

$$D_{\Gamma_\varepsilon} \subset U \times \mathcal{I}$$

and

$$H_2^D(U \times \mathcal{I}, \Lambda_\varepsilon''; \mathbb{R}) = H_2^D(\Sigma, L; \mathbb{R}) \oplus \mathbb{R} \cdot [D_{\Gamma_\varepsilon}],$$

where  $[D_{\Gamma_\varepsilon}]$  is the relative homology class of the disk  $D_{\Gamma_\varepsilon}$ . Since

$$H_2^D(W, K; \mathbb{R}) = 0$$

(recall that  $W$  is a tubular neighborhood of  $K$  in  $T^*K$ ), we have

$$\begin{aligned} H_2^D(U \times \mathcal{I} \times W, \Lambda_\varepsilon'; \mathbb{R}) &= H_2^D(U \times \mathcal{I}, \Lambda_\varepsilon''; \mathbb{R}) \oplus H_2^D(W, K; \mathbb{R}) = \\ &= H_2^D(U \times \mathcal{I}, \Lambda_\varepsilon''; \mathbb{R}) = H_2^D(U, L; \mathbb{R}) \oplus \mathbb{R} \cdot [D_{\Gamma_\varepsilon}]. \end{aligned}$$

Thus,

$$Im \nu_\sharp = \nu_\sharp(H_2^D(U, L; \mathbb{R})) + \mathbb{R} \cdot \nu_\sharp([D_{\Gamma_\varepsilon}]),$$

and since  $H_2^D(M, \Lambda_\varepsilon; \mathbb{R}) = Im \nu_\sharp + Q$ , we can write

$$H_2^D(M, \Lambda_\varepsilon; \mathbb{R}) = \nu_\sharp(H_2^D(U, L; \mathbb{R})) + \mathbb{R} \cdot \nu_\sharp([D_{\Gamma_\varepsilon}]) + Q.$$

Let us now pick an arbitrary  $A \in H_2^D(M, \Lambda_\varepsilon; \mathbb{R})$  and show that the integrals of  $\omega$  and  $dF \wedge dG$  over  $A$  are proportional with the proportionality coefficient being independent of  $A$ . There are three cases to consider:

1.  $A \in \nu_\sharp(H_2^D(U, L; \mathbb{R}))$ .
2.  $A = \nu_\sharp([D_{\Gamma_\varepsilon}])$ .
3.  $A \in Q$ .

Let us consider the first case:  $A = \nu_\sharp(A')$  for some  $A' \in H_2^D(U, L; \mathbb{R})$ . Let  $\eta$  be the symplectic form on  $T^*K$ . Since  $\nu^*\omega = d(s\lambda_0) \oplus \eta$  and  $\eta$  vanishes on  $H_2^D(U, L; \mathbb{R})$ , we get

$$\omega(A) = d(s\lambda_0)(A').$$

The classes in  $H_2^D(U, L; \mathbb{R})$  can be represented by disks  $(D^2, \partial D^2) \rightarrow (U \times s, L \times s) \subset U \times \mathcal{I}$  for some  $s \in \mathcal{I}$ ; accordingly,  $A$  can be represented by a disk whose boundary lies in  $\mathcal{H}$ . An easy application of the Stokes theorem yields that  $d(s\lambda_0)$  vanishes on  $H_2^D(U, L; \mathbb{R})$ , yielding  $\omega(A) = 0$ , and since  $FdG$  is zero on a neighborhood of  $\mathcal{H}$ , we also get  $dF \wedge dG(A) = 0$ .



Let us move to the second case:  $A = \nu_{\sharp}([D_{\Gamma_{\varepsilon}}])$ . Similarly to the above, we get

$$\omega(A) = d(s\lambda_0)([D_{\Gamma_{\varepsilon}}]).$$

A direct computation using the Stokes theorem shows that

$$\int_{D_{\Gamma_{\varepsilon}}} d(s\lambda_0) = (R_1 - R_0)T - \delta(\varepsilon),$$

that is,

$$\omega(A) = (R_1 - R_0)T - \delta(\varepsilon) =: C > 0.$$

At the same time another direct calculation using (16) yields

$$dF \wedge dG(A) = \nu^*(dF \wedge dG)([D_{\Gamma_{\varepsilon}}]) = \int_{\Gamma_{\varepsilon}} \nu^*(FdG) = -1.$$

Thus

$$dF \wedge dG(A) = -\omega(A)/C.$$

Finally, consider the third case:  $A \in Q$ . By the hypothesis of the proposition,  $\omega$  vanishes on  $Q$  and  $\partial(Q) \subset \Xi$ . Hence  $\omega(A) = 0$  and  $A$  can be represented by a disk whose boundary lies in  $\mathcal{H}$ . Since, by (16),  $FdG$  is zero on a neighborhood of  $\mathcal{H}$ , we get that

$$dF \wedge dG(A) = FdG(\partial A) = 0.$$

Thus, we have proved that

$$dF \wedge dG(A) = -\omega(A)/C$$

for any  $A \in H_2^D(M, \Lambda_{\varepsilon}; \mathbb{R})$ . This verifies the condition 2 of Proposition 5.1. Condition 3 follows from (17) and condition 1 is a part of the hypothesis of the proposition that we are probing. Therefore Proposition 5.1 can be applied and we get

$$\max_M \{F, G\} \geq 1/C = \left( (R_1 - R_0)T - \delta(\varepsilon) \right)^{-1}.$$

Since this is true for any  $F, G \in C_c^{\infty}(M)$  satisfying (16), we get

$$pb_4^{M,+}(\mathcal{F}_{\varepsilon}, \mathcal{C}_{\varepsilon}, \mathcal{L}_{\varepsilon}, \mathcal{H}_{\varepsilon}) \geq \left( (R_1 - R_0)T - \delta(\varepsilon) \right)^{-1},$$

where  $\delta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , as required.  $\square$

## 5.2 Spherical cotangent bundles

Below we describe a particular case where Proposition 5.2 can be applied.

Assume  $V^k$ ,  $k \geq 1$ , is a connected (not necessarily closed)  $k$ -dimensional manifold equipped with a complete Riemannian metric  $g$ . The metric  $g$  induces a norm on the cotangent bundle. Let  $\lambda$  be the canonical 1-form on  $T^*V$  and let  $d\lambda$  be the standard symplectic form on  $T^*V$ . Denote by  $\pi : T^*V \rightarrow V$  the natural projection.

We will apply the constructions and the results of Section 5.1 to the Lagrangian tetragons constructed in the following setting.

Let  $\Sigma$  be the unit cotangent bundle of  $V$ ,  $k \geq 2$  – it carries the standard contact structure defined by the contact form  $\lambda_0$  which is the restriction of  $\lambda$  to  $\Sigma$ . Let  $L$  be a closed connected Legendrian submanifold of  $\Sigma$  and let  $K = \mathbb{T}^m$  for some  $m \in \mathbb{Z}_{\geq 0}$ . If  $k = 1$ , the unit cotangent bundle of  $V$  has two connected components and we take  $\Sigma$  to be one of them.

Note that the Reeb flow  $\psi_t$  on  $\Sigma$  defined by  $\lambda_0$  is the geodesic flow of  $(V, g)$  which is well-defined for all  $t \in \mathbb{R}$  (in the case  $k = 1$  it is the geodesic flow in one of the two possible directions). Assume  $\psi_t(L)$  is disjoint from  $L$  for all  $t \in (0, T]$ . Fix  $0 < R_0 < R_1$ . Using this data define a Lagrangian tetragon  $\Lambda'' \subset \Sigma \times \mathbb{R}_+$ .

The symplectization  $(\Sigma \times \mathbb{R}_+, d(s\lambda_0))$  of  $\Sigma$  can be symplectically identified with the complement of the zero-section in  $T^*V$ : the symplectic embedding  $\nu' : (\Sigma \times \mathbb{R}_+, d(s\lambda_0)) \rightarrow (T^*V, d\lambda)$  sends each  $(v, s) \in \Sigma \times \mathbb{R}_+$  to  $sv \in T^*V$ . (Here we set  $\mathcal{I} = \mathbb{R}_+$ ). We identify  $\Lambda''$  with its image under  $\nu'$  and view  $\Lambda''$  as a Lagrangian tetragon in  $T^*V$ .

Let  $(M, \omega) := (T^*V \times T^*\mathbb{T}^m, d\lambda \oplus \eta)$ , where  $\eta$  is the standard symplectic form on  $T^*\mathbb{T}^m$ . Consider the symplectic embedding

$$\nu := \nu' \times Id : (\Sigma \times \mathbb{R}_+ \times T^*\mathbb{T}^m, d(s\lambda_0) \oplus \eta) \rightarrow (M, \omega) = (T^*V \times T^*\mathbb{T}^m, d\lambda \oplus \eta).$$

Let us define a Lagrangian tetragon

$$\Lambda = \mathcal{F} \cup \mathcal{C} \cup \mathcal{L} \cup \mathcal{H} := \Lambda'' \times K \subset M = T^*V \times T^*\mathbb{T}^m$$

and let  $\Lambda''_\epsilon$  and  $\Lambda_\epsilon = \Lambda''_\epsilon \times \mathbb{T}^m$  be the smoothenings of  $\Lambda''$  and  $\Lambda$  as in Section 5.1. (In the notation of Section 5.1 we identify  $\Lambda'$  and  $\Lambda$ ).

Let us see if Proposition 5.2 can be applied to the Lagrangian tetragon  $\Lambda$  in  $M$ .

One easily sees that  $H_2^D(M, \Lambda; \mathbb{R}) = \text{Im } \nu_\#$  if  $k \neq 2$  and for  $k = 2$  we have  $H_2^D(M, \Lambda; \mathbb{R}) = \text{Im } \nu_\# + Q$ , so that each class  $A \in Q$  can be represented by

a disk of the form  $D \times \text{point}$ , where  $\text{point}$  is the constant disk in  $\mathbb{T}^m$  and  $D$  is a disk in  $T^*V$  (crossing the zero-section) with boundary in  $R_0 \cdot L \subset T^*V$ . Thus, the boundary of  $D \times \text{point}$  lies in  $\mathcal{F}$  and therefore  $\partial A \in \Xi$ . Moreover,

$$\omega(A) = \int_{D \times \text{point}} \omega = \int_D d\lambda = \int_{\partial D} \lambda = \int_{\partial D} \lambda_0 = 0,$$

since  $\partial D \subset L$  and  $\lambda_0$  vanishes on the Legendrian submanifold  $L$ . Thus, the condition 1 of Proposition 5.2 is also satisfied.

Thus, in order to apply Proposition 5.2, it only remains to verify condition 2 of the proposition.

**Proposition 5.3.** *Assume that the image of the homomorphism  $\pi_1(L) \rightarrow \pi_1(V)$  induced by the projection  $\pi : L \rightarrow V$  is finite.*

*Then for any  $m \in \mathbb{Z}_{\geq 0}$  the symplectic manifold  $T^*V \times T^*\mathbb{T}^m$  admits no weakly exact Lagrangian submanifolds Lagrangian isotopic to  $\Lambda_\varepsilon$ .*

*In particular, for  $m = 0$  we get that the pair  $(\Sigma, L)$  is weakly non-exact.*

Therefore, Proposition 5.2 can be applied (for any  $m \in \mathbb{Z}_{\geq 0}$ ) yielding the following corollary.

**Corollary 5.4.** *Assume that the image of the homomorphism  $\pi_1(L) \rightarrow \pi_1(V)$  induced by the projection  $\pi : L \rightarrow V$  is finite.*

*Then*

$$\widehat{pb}_4^{T^*V \times T^*\mathbb{T}^m, +}(\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}) \geq \left( (R_1 - R_0)T \right)^{-1} > 0, \quad (18)$$

and thus, by Theorem 1.11, the Lagrangian tetragon  $\Lambda$  is stably  $(R_1 - R_0)T$ -interlinked in  $M = T^*V \times T^*\mathbb{T}^m$ .

Consequently, by (9), in the case  $K = \{\text{a point}\}$  the Lagrangian tetragon  $\Lambda$  is stably  $(R_1 - R_0)T$ -interlinked in  $\Sigma \times \mathbb{R}_+ \cong T^*V \setminus V$ .

**Proof of Proposition 5.3.** Assume, by contradiction, that  $Z \subset T^*V \times T^*\mathbb{T}^m$  is a weakly exact Lagrangian submanifold Lagrangian isotopic to  $\Lambda_\varepsilon$ .

Let  $\Pi : \tilde{V} \rightarrow V$  be the universal cover of  $V$ . It induces a covering  $\Pi^* : T^*\tilde{V} \rightarrow T^*V$  which in turn defines a covering  $\Pi^* \times Id : T^*(\tilde{V} \times \mathbb{T}^m) = T^*\tilde{V} \times T^*\mathbb{T}^m \rightarrow T^*V \times T^*\mathbb{T}^m$ . Denote by  $\tilde{\pi} : T^*\tilde{V} \rightarrow \tilde{V}$  the natural projection.

One easily sees that since the image of the homomorphism  $\pi_1(L) \rightarrow \pi_1(V)$  induced by the projection  $\pi : L \rightarrow V$  is finite,  $L$  lifts under  $\Pi^*$  to a

closed connected Legendrian submanifold  $\tilde{L}$  of the unit cotangent bundle of  $\tilde{V}$ . Therefore  $\Lambda_\varepsilon''$  lifts under  $\Pi^*$  to a closed connected Lagrangian submanifold  $\tilde{\Lambda}_\varepsilon'' \subset T^*\tilde{V}$  diffeomorphic to  $\tilde{L} \times \mathbb{S}^1$  and, accordingly,  $\Lambda_\varepsilon$  lifts under  $\Pi^* \times Id$  to a closed connected Lagrangian submanifold  $\tilde{\Lambda}_\varepsilon = \tilde{\Lambda}_\varepsilon'' \times \mathbb{T}^m \subset T^*\tilde{V} \times T^*\mathbb{T}^m$  diffeomorphic to  $\tilde{L} \times \mathbb{S}^1 \times \mathbb{T}^m$ . Since  $Z$  is Lagrangian isotopic to  $\Lambda_\varepsilon$  in  $T^*V \times T^*\mathbb{T}^m$ , it too lifts to a closed connected Lagrangian submanifold  $\tilde{Z}$  of  $T^*\tilde{V} \times T^*\mathbb{T}^m$  Lagrangian isotopic to  $\tilde{\Lambda}_\varepsilon$  and therefore diffeomorphic to  $\tilde{L} \times \mathbb{S}^1 \times \mathbb{T}^m$ . One easily sees that  $\tilde{Z}$  is weakly exact in  $T^*\tilde{V} \times T^*\mathbb{T}^m$ , because  $Z$  is weakly exact in  $T^*V \times T^*\mathbb{T}^m$ .

We claim that the degree of the projection of  $\tilde{Z} \subset T^*\tilde{V} \times T^*\mathbb{T}^m = T^*(\tilde{V} \times \mathbb{T}^m)$  to  $\tilde{V} \times \mathbb{T}^m$  is zero. (The degree is defined over  $\mathbb{Z}_2$ , since  $\tilde{Z}$  does not have to be orientable).

Indeed, note that the Lagrangian isotopy type of  $\Lambda_\varepsilon$  in  $T^*V \times T^*\mathbb{T}^m$  is independent of the parameters  $R_0, R_1, T$  (where  $\varepsilon$  is always chosen sufficiently small depending on  $R_0, R_1, T$ ). Thus, without loss of generality, we may assume that  $T$  is sufficiently small (depending on  $R_0$  and  $R_1$ ) so that  $\pi(\Lambda_\varepsilon'')$  lies in a small neighborhood of  $\pi(L)$  in  $V$ . Note that  $\pi(L) \subsetneq V$ . Therefore the projection  $\pi : \Lambda_\varepsilon'' \rightarrow V$  is not surjective. Hence the projection  $\pi \times Id : \Lambda_\varepsilon \rightarrow V \times \mathbb{T}^m$ , and hence the projection  $\tilde{\pi} \times Id : \tilde{\Lambda}_\varepsilon \rightarrow \tilde{V} \times \mathbb{T}^m$ , are also not surjective. Thus, the degree of  $\tilde{\pi} \times Id : \tilde{\Lambda}_\varepsilon \rightarrow \tilde{V} \times \mathbb{T}^m$  is zero, and since  $\tilde{\Lambda}_\varepsilon$  is Lagrangian isotopic to  $\tilde{Z}$ , so is the degree of  $\tilde{\pi} \times Id : \tilde{Z} \rightarrow \tilde{V} \times \mathbb{T}^m$ , proving the claim.

Now let us show that the weakly exact Lagrangian submanifold  $\tilde{Z} \subset T^*\tilde{V} \times T^*\mathbb{T}^m = T^*(\tilde{V} \times \mathbb{T}^m)$  is Lagrangian isotopic to an *exact* Lagrangian submanifold  $\bar{Z}$  of  $T^*(\tilde{V} \times \mathbb{T}^m)$  which would yield a contradiction, since it would violate Kragh's theorem [17]. (Recall that Kragh's theorem says that for a closed exact Lagrangian submanifold  $X$  of the cotangent bundle of an oriented manifold  $Y$  the projection  $X \rightarrow Y$  induces an isomorphism on homology, provided it induces an epimorphism  $\pi_1(X) \rightarrow \pi_1(Y)$ , and in particular, the degree of  $X \rightarrow Y$ , defined over  $\mathbb{Z}_2$ , has to be non-zero).

Let  $\tilde{\lambda}, \mu$  be the canonical 1-forms, respectively, on  $T^*\tilde{V}$  and  $T^*\mathbb{T}^m$ . Since  $\tilde{Z} \cong \tilde{L} \times \mathbb{S}^1 \times \mathbb{T}^m$ , we have  $H^1(\tilde{Z}; \mathbb{R}) \cong H^1(\tilde{L}; \mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R}^m$ . The components of the cohomology class  $[\tilde{\lambda} \oplus \mu|_{\tilde{Z}}] \in H^1(\tilde{Z}; \mathbb{R})$  corresponding to the first two summands are determined by integrals of  $\tilde{\lambda}$  over loops in  $\tilde{Z}$  that bound disks in  $T^*\tilde{V} \times T^*\mathbb{T}^m$  (the Lagrangian isotopy between  $\tilde{Z}$  and  $\tilde{\Lambda}_\varepsilon$  maps these loops in  $\tilde{Z}$  to loops in  $\tilde{\Lambda}_\varepsilon$  that are contractible in  $T^*\tilde{V} \times T^*\mathbb{T}^m$  because  $T^*\tilde{V}$  is simply connected). Therefore, since  $\tilde{Z}$  is weakly exact, these components of

$[\tilde{\lambda} \oplus \mu|_{\tilde{Z}}]$  are zero.

Thus  $[\tilde{\lambda} \oplus \mu|_{\tilde{Z}}]$  can be identified with a vector  $v \in \mathbb{R}^m$  lying in the  $\mathbb{R}^m$ -component of  $H^1(\tilde{Z}; \mathbb{R})$ . For each  $t \in [0, 1]$  define a symplectomorphism  $\Theta_t : T^*\mathbb{T}^m \rightarrow T^*\mathbb{T}^m$  by  $\Theta_t(p, q) := (p - tv, q)$ . Here  $p$  and  $q \bmod 1$  are the canonical Darboux coordinates on  $T^*\mathbb{T}^m$ . Then  $Id \times \Theta_t(\tilde{Z})$  is a Lagrangian isotopy and  $\bar{Z} := Id \times \Theta_1(\tilde{Z})$  is an exact Lagrangian submanifold which is Lagrangian isotopic to  $\tilde{Z}$ . This finishes the proof of the claim and of the proposition.  $\square$

As a corollary of the previous result we get Theorem 1.5.

**Proof of Theorem 1.5.** The Lagrangian tetragon  $\Lambda$  in  $T^*\mathbb{T}^k$  appearing in the statement of the theorem is exactly the Lagrangian tetragon constructed as above in the case where  $V = \mathbb{T}^k$ ,  $g$  is the Euclidean metric on  $\mathbb{T}^k$ ,  $L$  is a Legendrian fiber of  $\Sigma$ ,  $K = \{\text{a point}\}$  and  $0 < R_0 < R_1$ ,  $0 < T < 1/2$  are some numbers. Clearly, the image of  $\pi_1(L) \rightarrow \pi_1(\mathbb{T}^k)$  is trivial (hence, finite). Thus Corollary 5.4 can be applied yielding the theorem.  $\square$

**Remark 5.5.** In the case  $\dim V = k = 1$  and  $L = \{\text{a point}\} \subset \Sigma$ , one can actually prove that (18) is sharp, that is,

$$\widehat{pb}_4^{T^*V \times T^*K, +}(\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}) = \left( (R_1 - R_0)T \right)^{-1}. \quad (19)$$

Namely, in this case the four parts  $\mathcal{F}''$ ,  $\mathcal{C}''$ ,  $\mathcal{L}''$ ,  $\mathcal{H}''$  of the Lagrangian tetragon  $\Lambda'' := \mathcal{F}'' \cup \mathcal{C}'' \cup \mathcal{L}'' \cup \mathcal{H}'' \subset T^*V$  are given by

$$\mathcal{F}'' = [0, T] \times R_0, \quad \mathcal{C}'' = [0, T] \times R_1, \quad \mathcal{L}'' = T \times [R_0, R_1], \quad \mathcal{H}'' = 0 \times [R_0, R_1].$$

(Since  $V$  is a connected 1-dimensional manifold we may assume that either  $V = \mathbb{R}$  or  $V = \mathbb{S}^1$ ; if  $V = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ , we assume  $[0, T] \subset (0, 1) \subset \mathbb{S}^1 = V$ ). The Lagrangian tetragon  $\Lambda''$  bounds in  $T^*V$  a rectangle whose symplectic area is  $a := (R_1 - R_0)T$ . It is proved in [4] (see [7] for a minor correction of the proof in [4]) that  $pb_4(\mathcal{F}', \mathcal{C}', \mathcal{L}', \mathcal{H}') = 1/a$ . Combining this equality with (7), (8) and (18), we get

$$1/a = pb_4(\mathcal{F}', \mathcal{C}', \mathcal{L}', \mathcal{H}') \geq pb_4(\mathcal{F}' \times K, \mathcal{C}' \times K, \mathcal{L}' \times K, \mathcal{H}' \times K) =$$

$$\begin{aligned}
&= pb_4(\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}) \geq pb_4^+(\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}) \geq \\
&\geq \widehat{pb}_4^+(\mathcal{F}, \mathcal{C}, \mathcal{L}, \mathcal{H}) \geq 1/a = \left( (R_1 - R_0)(r_1 - r_0) \right)^{-1},
\end{aligned}$$

yielding (19).

In view of Theorem 1.11 this means that  $(R_1 - R_0)T$  is the smallest  $\kappa$  for which the Lagrangian tetragon  $\Lambda$  is  $\kappa$ -interlinked.

### 5.3 Symplectic fillings

Here is another class of examples where Proposition 5.2 can be applied.

Let us recall a few basic notions related to symplectic fillings of contact manifolds (see e.g. [12], [13]). Assume  $(\Sigma, \xi = \ker \lambda_0)$  is a closed, not necessarily connected, contact manifold which is the boundary of a compact symplectic manifold  $(M, \omega)$ . Assume that on a neighborhood of  $\Sigma = \partial M$  in  $M$  there exists a vector field  $v$  such that  $L_v \omega = \omega$  (a vector field satisfying this property is called a Liouville vector field) and  $i_v \omega|_\Sigma$  is a contact form for  $\xi$  defining the same coorientation of  $\xi$  as  $\lambda_0$ . In such a case  $(M, \omega)$  is called a *strong symplectic filling* of  $(\Sigma, \xi)$  and  $(\Sigma, \xi)$  is called *strongly symplectically fillable*. It is easy to show that a collar neighborhood of  $\Sigma$  in such an  $M$  is symplectomorphic to  $(\Sigma \times (1 - \epsilon, 1], d(s\lambda_0))$  for some small  $\epsilon > 0$  (here  $s$  is the coordinate on  $(1 - \epsilon, 1]$ ). Thus  $M$  can be completed to an open symplectic manifold  $(\bar{M}, \bar{\omega}) := (M, \omega) \cup (\Sigma \times (1 - \epsilon, +\infty), d(s\lambda_0))$  without boundary which at infinity coincides with the symplectization of  $\Sigma$  and thus is of bounded geometry at infinity [3], [11].

If  $(\Sigma, \xi = \ker \lambda_0)$  is only a union of some of the connected components of a contact manifold admitting a strong symplectic filling  $(M, \omega)$ , then  $(\Sigma, \xi = \ker \lambda_0)$  is called *strongly symplectically semi-fillable* and  $(M, \omega)$  is called a *strong symplectic semi-filling*<sup>3</sup> of  $(\Sigma, \xi = \ker \lambda_0)$ .

Certain contact manifolds admit strong symplectic (semi)-fillings  $(M, \omega)$  which are *subcritical Weinstein domains* – recall (see e.g. [8]) that the Weinstein domain structure on  $(M, \omega)$  is a pair  $(v, f)$  consisting of a smooth function  $f$  with only non-degenerate and birth-death critical points and such that  $\partial M$  is its regular level set, and a Liouville vector field  $v$  on  $M$  (with respect to  $\omega$ ) which is gradient-like with respect to  $f$ . A Weinstein domain is

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<sup>3</sup>As it was shown in [12], [13], in the case  $\dim \Sigma = 3$  strong symplectic *semi-fillability* is equivalent to strong symplectic *fillability*.

called *subcritical* if all critical points of  $f$  have index  $< n$ , where  $2n = \dim M$ . If  $M$  is a subcritical Weinstein domain, then its completion  $(\bar{M}, \bar{\omega})$  is a *subcritical Weinstein manifold*, meaning that  $M$  admits a pair  $(\bar{v}, \bar{f})$ , with  $\bar{v}$  being a complete Liouville vector field (with respect to  $\bar{\omega}$ ) and  $\bar{f}$  is a proper smooth function bounded from below which is gradient-like with respect to  $\bar{v}$  and has only non-degenerate critical points of index  $< n$  or birth-death critical points (see [8]). The key property of subcritical Weinstein manifolds is that any compact set in such a symplectic manifold is displaceable (see [5], Lemma 3.2, where this claim is proved for subcritical Stein manifolds – the analog of Weinstein manifolds in the holomorphic setting; the proof for Weinstein manifolds is the same). A subcritical Weinstein manifold (or more generally, any Weinstein manifold) is exact (see [8]).

**Example 5.6.** The unit sphere  $\Sigma = \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$ ,  $n \geq 2$ , with the standard contact structure admits a strong symplectic filling  $M$  – the unit ball in  $\mathbb{R}^{2n}(p, q)$  with the standard symplectic structure  $dp \wedge dq$  – which is a subcritical Weinstein domain: in this case (in the vector notation)  $v = p\partial/\partial p + q\partial/\partial q$  and  $f(p, q) = p^2 + q^2$ . The corresponding Weinstein manifold  $\bar{M}$  is the standard symplectic  $\mathbb{R}^{2n}$ .

Having finished the review, let us assume that  $(\Sigma, \xi = \ker \lambda_0)$  is a closed connected contact manifold admitting a strong symplectic semi-filling  $(M, \omega)$ . Let  $\psi_t : \Sigma \rightarrow \Sigma$  be the Reeb flow of  $\lambda_0$ . Let  $L$  be a closed connected Legendrian submanifold of  $(\Sigma, \xi)$  and pick  $T > 0$  so that  $\psi_t(L) \cap L = \emptyset$  for any  $t \in (0, T]$ . Fix some  $0 < R_0 < R_1$ . Let  $K = \{\text{a point}\}$ . Using this data define a Lagrangian tetragon  $\Lambda \subset (\Sigma \times \mathbb{R}_+, d(s\lambda_0))$ . Let  $\nu : (\Sigma \times (1, +\infty), d(s\lambda_0)) \rightarrow (\bar{M}, \bar{\omega})$  be the inclusion. If  $R_0 > 1$ , the image of  $\Lambda$  under  $\nu$  is well-defined – it is a Lagrangian tetragon in  $(\bar{M}, \bar{\omega})$  that will be also denoted by  $\Lambda$ .

We are going to apply Proposition 5.2 to stabilizations of  $\Lambda$ , where  $\Lambda$  is viewed as a tetragon in  $(\Sigma \times \mathbb{R}_+, d(s\lambda_0))$  or in  $(\bar{M}, \bar{\omega})$ . The following theorems list possible conditions under which this can be done.

**Theorem 5.7.** *Assume that the symplectic semi-filling  $(M, \omega)$  of  $(\Sigma, \xi)$  is symplectically aspherical<sup>4</sup> and the homomorphism  $\pi_1(L) \rightarrow \pi_1(\Sigma)$  is trivial.*

*Then  $\Lambda$  is stably  $(R_1 - R_0)T$ -interlinked in  $(\bar{M}, \bar{\omega})$ .*

**Theorem 5.8.** *Assume that the symplectic semi-filling  $(M, \omega)$  of  $(\Sigma, \xi)$  is a subcritical Weinstein domain and the homomorphism  $\pi_1(\Sigma) \rightarrow \pi_1(M)$  is injective.*

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<sup>4</sup>One easily sees that this is equivalent to  $(\bar{M}, \bar{\omega})$  being symplectically aspherical.

Then  $\Lambda$  is stably  $(R_1 - R_0)T$ -interlinked in  $(\bar{M}, \bar{\omega})$ .

The proofs of Theorems 5.7, 5.8 are given further in this section. They are based on the following propositions that will be proved in Section 5.4.

**Proposition 5.9.** *Assume that the symplectic semi-filling  $(M, \omega)$  of  $(\Sigma, \xi)$  is symplectically aspherical and the homomorphism  $\pi_1(L) \rightarrow \pi_1(\Sigma)$  is trivial. Let  $m \in \mathbb{Z}_{\geq 0}$  and let  $p, q$  be the standard Darboux coordinates on  $T^*\mathbb{T}^m$ .*

*Then the symplectic manifold  $(\Sigma \times \mathbb{R}_+ \times T^*\mathbb{T}^m, d(s\lambda_0 + pdq))$  admits no closed weakly exact Lagrangian submanifolds Lagrangian isotopic to  $\Lambda_\varepsilon \times \mathbb{T}^m$ .*

*In particular, for  $m = 0$ , the pair  $(\Sigma, L)$  is weakly non-exact.*

**Proposition 5.10.** *Assume that the symplectic semi-filling  $(M, \omega)$  of  $(\Sigma, \xi)$  is a subcritical Weinstein domain.*

*Then for any  $m \in \mathbb{Z}_{\geq 0}$  the symplectic manifold  $\bar{M} \times T^*\mathbb{T}^m$  admits no closed weakly exact Lagrangian submanifolds.*

*In particular, for  $m = 0$ , we get that for any closed Legendrian submanifold  $L \subset (\Sigma, \xi)$  the pair  $(\Sigma, L)$  is weakly non-exact.*

**Proof of Theorem 5.7.** The theorem follows from Proposition 5.2 applied to the  $\mathbb{T}^m$ -stabilization of  $\Lambda$ . Indeed, let us verify the two conditions of Proposition 5.2 in this case.

Since the homomorphism  $\pi_1(L) \rightarrow \pi_1(\Sigma)$  is trivial, the map

$$H_2^D(\Sigma, L; \mathbb{R}) \rightarrow H_2^D(\bar{M}, \Lambda; \mathbb{R})$$

is surjective and therefore

$$H_2^D(\bar{M} \times T^*\mathbb{T}^m, \Lambda \times \mathbb{T}^m; \mathbb{R}) = \text{Im } \nu'_\# + Q,$$

where  $\nu'_\#$  is the map on homology induced by  $\nu' := \nu \times \text{Id} : \Sigma \times \mathbb{R}_+ \times T^*\mathbb{T}^m \rightarrow \bar{M} \times T^*\mathbb{T}^m$  and  $Q = H_2^S(\bar{M} \times T^*\mathbb{T}^m; \mathbb{R})$ . Obviously,  $\partial(Q) = 0 \in \Xi$  and, since  $(\bar{M}, \bar{\omega})$  is assumed to be symplectically aspherical,  $\bar{\omega}|_Q = 0$ . Thus, condition 1 of Proposition 5.2 holds.

Condition 2 of Proposition 5.2 is stated in Proposition 5.9.

Thus, Proposition 5.2 can be applied, yielding the theorem.  $\square$

**Proof of Theorem 5.8.** The theorem follows from Proposition 5.2 applied to the  $\mathbb{T}^m$ -stabilization of  $\Lambda$ . Indeed, let us verify the two conditions of Proposition 5.2 in this case.



Let us check condition 1 of that proposition. Since the homomorphism  $\pi_1(\Sigma) \rightarrow \pi_1(M)$  is injective, so is the homomorphism

$$\pi_1(\Sigma \times (1, +\infty) \times T^*\mathbb{T}^m) \rightarrow \pi_1(\bar{M} \times T^*\mathbb{T}^m).$$

Hence, the map

$$H_2^D(\Sigma, L; \mathbb{R}) \rightarrow H_2^D(\bar{M}, \Lambda; \mathbb{R})$$

is surjective and therefore

$$\begin{aligned} H_2^D(\bar{M} \times T^*\mathbb{T}^m, \Lambda \times T^*\mathbb{T}^m; \mathbb{R}) &= \\ &= H_2^D(\Sigma \times (1, +\infty) \times T^*\mathbb{T}^m, \Lambda \times \mathbb{T}^m; \mathbb{R}) + \\ &+ \text{Im} \left( H_2^S(\bar{M} \times T^*\mathbb{T}^m; \mathbb{R}) \rightarrow H_2^D(\bar{M} \times T^*\mathbb{T}^m, \Lambda \times \mathbb{T}^m; \mathbb{R}) \right). \end{aligned}$$

Since  $\bar{\omega}$  is exact, the symplectic form  $\bar{\omega} + dp \wedge dq$  (where  $p, q$  are the standard Darboux coordinates on  $T^*\mathbb{T}^m$ ) vanishes on  $\text{Im} (H_2^S(\bar{M}; \mathbb{R}) \rightarrow H_2^D(\bar{M}, \Lambda; \mathbb{R}))$ . This means that condition 1 of Proposition 5.2 is satisfied.

Condition 2 of Proposition 5.2 has been verified in Proposition 5.10.

Thus, Proposition 5.2 can be applied and the theorem follows.  $\square$

**Proof of Theorem 1.7.** The needed result follows immediately from Theorem 5.8 and Example 5.6.  $\square$

## 5.4 Non-existence of weakly exact Lagrangian submanifolds

In this section we will prove Proposition 5.9 and Proposition 5.10.

Proposition 5.9 will be deduced from the following somewhat stronger statement.

**Proposition 5.11.** *Assume that the symplectic semi-filling  $(M, \omega)$  of  $(\Sigma, \xi)$  is symplectically aspherical. Let  $m \in \mathbb{Z}_{\geq 0}$  and let  $p, q$  be the standard Darboux coordinates on  $T^*\mathbb{T}^m$ . Set  $X := \Sigma \times \mathbb{R}_+ \times T^*\mathbb{T}^m$ ,  $\theta := s\lambda_0 + pdq$ .*

*Then the symplectic manifold  $(X, d\theta)$  admits no closed Lagrangian submanifolds  $Z$  with the following property:  $H_1(Z; \mathbb{R})$  admits a split  $H_1(Z; \mathbb{R}) = S \oplus T$  such that  $\theta|_S \equiv 0$  and the restrictions to  $S$  and  $T$  of the homomorphism  $H_1(Z; \mathbb{R}) \rightarrow H_1(\mathbb{T}^m; \mathbb{R})$ , induced by the natural projection, are, respectively, the zero map and an isomorphism:  $T \cong H_1(\mathbb{T}^m; \mathbb{R})$ .*

**Proof of Proposition 5.11.** Assume by contradiction that  $Z$  is a closed weakly exact Lagrangian submanifold of  $(X, d\theta)$  satisfying the condition listed in Proposition 5.11 above.

Denote the elements of  $T \subset H_1(Z; \mathbb{R})$  mapped by the isomorphism  $T \rightarrow H_1(\mathbb{T}^m; \mathbb{R})$  to the standard generators of  $H_1(\mathbb{T}^m; \mathbb{R})$  by  $e_1, \dots, e_m$ . Set

$$a_i := \int_{e_i} \theta, \quad i = 1, \dots, m,$$

and

$$a := (a_1, \dots, a_m).$$

Consider a symplectomorphism  $\varphi$  of  $X = \Sigma \times \mathbb{R}_+ \times T^*\mathbb{T}^m$  of the form

$$\varphi(x, s, p, q) = (x, s, p - a, q).$$

It maps  $Z$  to a closed weakly exact Lagrangian submanifold  $Z' \subset X$ . Write

$$H_1(Z'; \mathbb{R}) = \varphi_*(S) \oplus \varphi_*(T).$$

Note that

$$\varphi^*\theta = \varphi^*(s\lambda_0 + pdq) = s\lambda_0 + pdq - adq = \theta - adq. \quad (20)$$

We claim that  $Z'$  is not only weakly exact but actually *exact*. Indeed, let  $A \in H_1(Z'; \mathbb{R})$  and let us check that  $\int_A \theta = 0$ .

If  $A \in \varphi_*(S)$ , then  $A = \varphi_*(\tilde{A})$  for some  $\tilde{A} \in S$ . Hence,

$$\int_A \theta = \int_{\tilde{A}} \varphi^*\theta = \int_{\tilde{A}} \theta - adq = 0,$$

since, by the assumption of the proposition, both  $\theta$  and  $adq$  vanish on  $S$ .

Let  $A = \varphi_*(e_i) \in \varphi_*(T)$  for some  $i = 1, \dots, m$ . Since  $\varphi$  is isotopic to the identity, the image of  $\varphi_*(e_i)$  under the  $H_1(Z'; \mathbb{R}) \rightarrow H_1(X; \mathbb{R})$  coincides with the image of  $e_i$  under  $H_1(Z; \mathbb{R}) \rightarrow H_1(X; \mathbb{R})$ , that is,  $\varphi_*(e_i)$  is the  $i$ -th standard generator of  $H_1(\mathbb{T}^m; \mathbb{R}) \subset H_1(X; \mathbb{R})$ . Therefore  $\int_{\varphi_*(e_i)} dq_j = 1$  if  $i = j$  and 0 otherwise. Also Combining these observations we get

$$\int_A \theta = \int_{e_i} \varphi^*\theta = \int_{e_i} \theta - adq =$$

$$= \int_{e_i} \theta - \int_{e_i} adq = a_i - a_i = 0,$$

finishing the proof of the claim that  $Z'$  is exact.

The vector field  $s\partial/\partial s \oplus p\partial/\partial p$  on  $(X = \Sigma \times \mathbb{R}_+ \times T^*\mathbb{T}^m, d\theta)$  is Liouville and its flow  $\phi_t$  for any  $t > 0$  is a conformal symplectomorphism. Since  $Z'$  is exact, the family  $\{\phi_t(Z')\}_{t_0 \leq t \leq t_1}$  is an exact Lagrangian isotopy and thus can be extended to an ambient Hamiltonian isotopy of  $(X, d\theta)$ . The flow  $\phi_t$  is just the multiplication by  $e^t$  in the  $\mathbb{R}_+$  and  $\mathbb{R}^m$ -factors of  $X = \Sigma \times \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{T}^m$  and thus for a sufficiently large  $t > 0$  it moves any compact set inside  $\Sigma \times (1, +\infty) \times T^*\mathbb{T}^m$  and displaces it from itself.

Thus, if  $t > 0$  is sufficiently large,  $\phi_t(Z')$  is an exact displaceable Lagrangian submanifold of  $(X, \theta)$ . Since  $\Sigma \times (1, +\infty) \subset \bar{M}$ , the exact Lagrangian submanifold  $\phi_t(Z') \subset X = \Sigma \times \mathbb{R}_+ \times T^*\mathbb{T}^m \subset \bar{M} \times T^*\mathbb{T}^m$  is also displaceable in a symplectic manifold  $\bar{M} \times T^*\mathbb{T}^m$  of bounded geometry at infinity. This is impossible by a theorem of Gromov [15], [3]. Thus, we have obtained a contradiction, meaning that  $(X, d\theta)$  admits no compact weakly exact Lagrangian submanifolds satisfying the condition as in the statement of the proposition.  $\square$

**Proof of Proposition 5.9.** As before, denote  $X := \Sigma \times \mathbb{R}_+ \times T^*\mathbb{T}^m$  and  $\theta := s\lambda_0 + pdq$  (the 1-form on  $X$ ).

Assume by contradiction that  $Z$  is a weakly exact Lagrangian submanifold of  $(X, d\theta)$  Lagrangian isotopic to  $\Lambda_\varepsilon \times \mathbb{T}^m$ . Let  $I : H_1(\Lambda_\varepsilon \times \mathbb{T}^m; \mathbb{R}) \rightarrow H_1(Z; \mathbb{R})$  be the induced isomorphism on homology. Split  $H_1(Z; \mathbb{R})$  as

$$H_1(Z; \mathbb{R}) = S + T,$$

where

$$S := I(H_1(\Lambda_\varepsilon; \mathbb{R}))$$

and

$$T := I(H_1(\mathbb{T}^m; \mathbb{R})).$$

Since  $Z$  is Lagrangian isotopic to  $\Lambda_\varepsilon \times \mathbb{T}^m$ , the restrictions of the homomorphism  $H_1(Z; \mathbb{R}) \rightarrow H_1(\mathbb{T}^m; \mathbb{R})$ , induced by the natural projection, to  $S$  and  $T$  are, respectively, the zero map and an isomorphism.

It follows from the construction of a (smoothened) Lagrangian tetragon that, since the homomorphism  $\pi_1(L) \rightarrow \pi_1(\Sigma)$  is trivial, so is the homomorphism  $\pi_1(\Lambda_\varepsilon) \rightarrow \pi_1(\Sigma \times \mathbb{R}_+)$ . Hence,  $S$  lies in the image of the boundary

homomorphism  $\partial : H_2^D(X, Z; \mathbb{R}) \rightarrow H_1(Z; \mathbb{R})$ . Since  $Z$  is weakly exact, it means that  $\theta|_S = 0$ .

Consider the homomorphism

$$H_1(\Lambda_\varepsilon \times \mathbb{T}^m; \mathbb{R}) = H_1(\Lambda_\varepsilon; \mathbb{R}) \oplus H_1(\mathbb{T}^m; \mathbb{R}) \rightarrow H_1(\mathbb{T}^m; \mathbb{R})$$

induced by the natural projection. Its restriction to  $H_1(\mathbb{T}^m; \mathbb{R})$  is, obviously, an isomorphism. Since  $Z$  is isotopic to  $\Lambda_\varepsilon \times \mathbb{T}^m$ , the restriction to  $T$  of the homomorphism

$$H_1(Z; \mathbb{R}) = S + T \rightarrow H_1(\mathbb{T}^m; \mathbb{R}),$$

induced by the natural projection, is also an isomorphism.

Thus,  $Z$  satisfies the conditions listed in Proposition 5.11 which says that no such  $Z$  exist. This yields a contradiction which finishes the proof.  $\square$

**Proof of Proposition 5.10.** Let  $m \in \mathbb{Z}_{\geq 0}$ . Assume  $Z$  is a closed weakly exact Lagrangian submanifold of  $\bar{M} \times T^*\mathbb{T}^m$ . Since  $(\bar{M}, \bar{\omega})$  is a subcritical Weinstein manifold,  $\bar{M} \times T^*\mathbb{T}^m$  is of bounded geometry at infinity and exact and, in addition, any compact set is displaceable in  $(\bar{M}, \bar{\omega})$ . The latter condition implies that in  $\bar{M} \times T^*\mathbb{T}^m$  too any compact set is displaceable. Thus,  $Z$  is displaceable in  $\bar{M} \times T^*\mathbb{T}^m$ . This violates the theorem of Gromov [15], [3] cited above, yielding a contradiction. Hence,  $\bar{M} \times T^*\mathbb{T}^m$  admits no closed weakly exact Lagrangian submanifolds.  $\square$

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